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Reduction of Representations of SU_{mn} with respect to the Subgroup $SU_m \otimes SU_n$

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With a view to application to theories such as the Wigner supermultiplet theory of nuclear structure and its recent extension into the domain of particle physics, the reduction of representations of the special unitary group SU_{mn} into representations of its $SU_m \otimes SU_n$ subgroup is considered. Proof is given of the result, that, in the case of a representation of SU_{mn} of plurality type p_{mn}, this reduction yields only those irreducible representations of $SU_m \otimes SU_n$, whose SU_m and SU_n parts have plurality types p_m and p_n equal to p_{mn} modulo m and modulo n, respectively. A practical method of obtaining reductions is described and a tabulation of results in the case of SU_6 and $SU_2 \otimes SU_3$ is made. The method depends on the use of a very concise version of the Weyl character formula for unitary groups which does not seem to have been used elsewhere.

I. INTRODUCTION

MUCH work during the last year has been performed in investigation of the possibility that the octet version[1] of the SU_3 theory of particle interactions is only a manifestation of some wider theory described by an approximate symmetry group of which SU_3 is a subgroup. Such work divides quite clearly into three distinct areas. In one of these areas, the extended symmetry group in question refers only to internal space properties of particles, and the most natural candidates in this case are the groups of rank three. Of these the group SU_4 seems to be the most popular[2] though various others have also been investigated.[3] A second area of activity concerns the extension[4] of the Wigner supermultiplet theory of nuclear structure[5] into the realm of particle physics. Here the underlying symmetry is that described by the group SU_6 with an

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[1] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961); M. Gell-Mann, California Institute of Technology Report CTSL–20, 1961, (unpublished) and Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[2] P. Tarjanne and V. Teplitz, Phys. Rev. Letters **11**, 447 (1963); D. E. Neville, *ibid.* **13**, 118 (1964); Y. Hara, Phys. Rev. **134**, B701 (1964); H. Bacry, J. Nuyts, and J. Prentki, Phys. Letters **9**, 279 (1964); C. R. Hagen and A. J. Macfarlane, Phys. Rev. **135**, B432 (1964); W. Krolikowski, Nuovo Cimento **33**, 243 (1964), and Nucl. Phys. **52**, 342 (1964); L. Cohen, preprint (1964); D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Phys. Letters **11**, 100 (1964); J. D. Bjorken and S. L. Glashow, *ibid.*, 255; I. S. Gerstein and M. L. Whippman, Phys. Rev. **136**, B829 (1964) and preprint, 1964; D. Speiser and R. J. Oakes, Phys. Rev. Letters **13**, 579 (1964); M. Parkinson, *ibid.*, 588; Z. Maki and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **32**, 144 (1964); V. V. Vladimirski, preprint (1964). See also A. Salam, International Center for Theoretical Physics, Trieste, unpublished report where references may be found to related contributions to the 1964 International Conference on High Energy Physics, Dubna, August 1964.

[3] C. Lovelace, Nuovo Cimento **37**, 225(1965); H. Bacry, J. Nuyts, and J. Prentki, Ref. 2 and preprint (1964); I. S. Gerstein and M. L. Whippman, preprint, Ref. 2; G. Costa, Nuovo Cimento **34**, 257 (1964). See also A. Salam, Ref. 2.

[4] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.*, 175; F. Gürsey, A. Pais, and L. A. Radicati, *ibid.*, 299; T. K. Kuo and T. Yao, *ibid.*, 415; M. A. B. Beg, B. W. Lee, and A. Pais, *ibid.*, 514; M. A. B. Beg and V. Singh, *ibid.*, 418 and 509; B. Sakita, *ibid.*, 643 (1964); and Phys. Rev. H. J. Lipkin, *ibid.*, 590.

[5] E. P. Wigner, Phys. Rev. **51**, 106 (1937); E. P. Wigner nad E. Feenberg, Rept. Progr. Phys. **8**, 274 (1941).

$SU_2 \otimes SU_3$ subgroup in which the SU_2 part refers to ordinary spin and the SU_3 part refers in the sense of the octet theory to the internal space properties of particles. The third area is associated with the $U_3 \otimes U_3$ group.[6]

In connection with their work on the first of the above-mentioned areas of research, the authors[7] made a study of certain properties of groups of the type SU_{n+1}. In this study the concept of purality type[8] for representations of SU_{n+1} was introduced together with theorems concerning the reduction of a given representation of SU_{n+1} into representations of its $SU_n \otimes U_1^{(n)}$ subgroup. Such theorems are important in connection with the classification of particles according to basis states of representations of SU_3, SU_4, etc. In addition they allow the occurrence of fractional eigenvalues of the generator $Y_1^{(n)}$ of $U_1^{(n)}$ (and, hence, of Q/e where Q is the charge operator[7,9]) to be correlated with the plurality type of a given representation of SU_{n+1}.

In the present paper we study certain properties of special unitary groups relevant to the second category of theories referred to in the first paragraph. Most of the results of this study concern the reduction of representations of SU_{mn} into representations of its subgroup $SU_m \otimes SU_n$. To be more specific, we must introduce some notation. The irreducible representations (IRs) of SU_r can be put in one to one correspondence with Young diagrams of no more than $(r-1)$ rows and, hence, characterized by the notation[10]

$$\{l_1, l_2, \cdots l_{r-1}\}, \tag{1.1}$$

where l_i gives the number of boxes in the ith row of the associated Young diagram and satisfies

$$l_1 \geq l_2 \geq \cdots \geq l_{r-1} \geq 0. \tag{1.2}$$

The plurality type of the IR $\{l_1, l_2, \cdots l_{r-1}\}$ is defined[7] according to

$$p\{l_1, l_2, \cdots l_{r-1}\} = \sum_{i=1}^{r-1} l_i \text{ (modulo } r). \tag{1.3}$$

Its basic property[7] is contained in the observation that an IR of SU_r which occurs in the direct product of a given pair of IRs has plurality type equal, modulo r, to the sum of the plurality types of the given pair of IRs. Our discussion begins with a proof (in Sec. II) of the following

Theorem. The reduction of an IR of SU_{mn} with plurality type p_{mn} into IRs of its $SU_m \otimes SU_n$ subgroup yields only IRs in which

(a) the SU_m part has plurality type p_m given by $p_m = p_{mn}$ (modulo m);
(b) the SU_n part has plurality type p_n given by $p_n = p_{mn}$ (modulo n).

Well known results[4] for SU_6 illustrate this theorem. For example, the 56-dimensional IR $\{3, 0, 0, 0, 0\}$ (or $\{3\}$ simply) of SU_6 has "hexality" 3 and, hence, by the theorem should contain only IRs of $SU_2 \otimes SU_3$ with (duality, triality) $= (1, 0)$. In fact it contains a spin-$\frac{3}{2}$ decuplet and a spin-$\frac{1}{2}$ octet, both of which have the required duality–triality values. The theorem may also be used to make statements of the following type:

(a) a theory with symmetry group SU_6/Z_6, i.e., a theory generated by the regular representation $\{2, 1, 1, 1, 1\}$ of 35 dimensions of SU_6 describes only unquarklike bosons;

(b) a theory with symmetry group SU_6/Z_3, i.e., a theory generated by the fundamental IR $\{1, 1, 1, 0, 0\}$ (or simply $\{1^3\}$) of SU_6, describes only unquarklike bosons and fermions;

(c) a theory with symmetry group SU_6/Z_2, i.e., a theory generated by the complex conjugate pair of fundamental IRs $\{1^2\}$ and $\{1^4\}$ of SU_6, describes only bosons both quarklike and unquarklike;

(d) a theory with symmetry group SU_6 itself, i.e., a theory generated by all five fundamental IRs $\{1^r\}$ $(1 \leq r \leq 5)$, describes quarklike and unquarklike bosons and fermions.

The term unquarklike is used here to designate particles associated with the IRs of SU_3 which occur in the octet theory, i.e., the theory whose symmetry group is SU_3/Z_3, the term quark having been introduced[11] to describe the particles associated with the fundamental IR $\{1\}$ of SU_3.

[6] R. E. Marshak and S. Okubo, Nuovo Cimento **19**, 1226 (1961); J. Schwinger, Phys. Rev. Letters **12**, 237 (1964); Phys. Rev. **135**, B816 (1964); Phys. Rev. Letters **13**, 355 (1964); and Phys. Rev. **136**, B1821 (1964); I. S. Gerstein and K. T. Mahanthappa, Phys. Rev. Letters **12**, 570 (1964), and Nuovo Cimento **36**, 488 (1965); M. Suzuki, Phys. Rev. **136**, B272 (1964); E. Johnson and R. F. Sawyer, Phys. Letters **9**, 212 (1964); E. Johnson, R. F. Sawyer, and K. C. Wali, Phys. Rev. Letters **13**, 141 (1964); P. G. O. Freund and Y. Nambu, *ibid.* p. 220; *ibid.* **12**, 714 (1964); M. Gell-Mann, Physics **1**, 63 (1964); F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. **135**, B467 (1964); R. E. Marshak, N. Mukunda, and S. Okubo, Phys. Rev. **137**, B698 (1965). See also A. Salam, Ref. 2.

[7] C. R. Hagen and A. J. Macfarlane, J. Math. Phys. **5**, 1335 (1964).

[8] The corresponding concept for SU_3 was first discussed by L. C. Biedenharn and E. C. Fowler, Nuovo Cimento **33**, 1329 (1964).

[9] I. S. Gerstein, J. Math. Phys. **6**, 469 (1965).

[10] Connection with the highest notation is as follows.[7] The highest weight notation for $\{l_1, l_2, \cdots l_{r-1}\}$ is $(\lambda_1, \lambda_2, \cdots \lambda_{r-1})$ with $\lambda_1 = l_1 - l_2$, $\lambda_2 = l_2 - l_3$, $\cdots$, $\lambda_{r-1} = l_{r-1}$. The allowed λ_i are $\lambda_i \geq 0$ $(1 \leq i \leq r-1)$ in agreement with Eq. (1.2).

[11] M. Gell-Mann, Phys. Letters **8**, 214 (1964).

The statements (a) to (d) follow from the theorem in a straightforward way. Consider (b), for example. Since only the IRs $\{1^2\}$ and $\{1^4\}$ of SU_6 with even hexality are used to generate the IRs of the theory based on SU_6/Z_2, it follows from the basic property of plurality type quoted above that the theory involves only IRs of even hexality. The theorem then implies that IRs of $SU_2 \otimes SU_3$ which are contained in such IRs have duality zero and thus correspond only to bosons. However, since hexalities 0, 2, 4 are allowed, all values of triality type can occur for the SU_3 parts of the IRs of $SU_2 \otimes SU_3$.

Statement (b) is of some importance, since it implies that the extensions of supermultiplet theory to particle physics can be made whether quarks are discovered or not. Of course, such a statement is already implicit in the literature.[5]

The theorem proved in Sec. II is concerned with the reduction of an IR of SU_{mn} into IRs of $SU_m \otimes SU_n$. This leads us naturally to consider the development of a rapid and economical method of obtaining such reductions. The basic idea is to express the character of a given IR of SU_{mn} corresponding to an element of the $SU_m \otimes SU_n$ subgroup as a sum of characters of IRs of the product subgroup. In order to make the application of this fundamental idea a practical matter, we employ, not the Weyl character formula[12] itself, but certain equivalent more concise forms of it obtained by algebraic manipulation. Discussion of this character formula is given in Sec. III. In Sec. IV, technical details are given for the use of these formulas to derive the desired results. General results have not been obtained, but rather a method to use in any particular example. Accordingly, the technique is explained in terms of the supermultiplet theory, as the simplest example available, as well as its particle-physics counterpart. In the former instance, results are well known[13] for a wide range of low-dimensional IRs of SU_4, and hence, we have not repeated them. For the case of SU_6, we have appended a table of the low-dimensional IRs, their dimensionalities and hexalities, their $SU_2 \otimes SU_3$ content and the duality–triality of this content.

II. PROOF OF THE REDUCTION THEOREM

We prove here the following

Theorem. The reduction of an IR of SU_{mn} of plurality p_{mn} into IRs of $SU_m \otimes SU_n$ yields only IRs whose SU_m and SU_n parts have pluralities p_m and p_n equal to p_{mn}, modulo m and modulo n, respectively.

The method of proof is to establish the theorem for each of the fundamental IRs of SU_{mn}. Then, from the basic property of plurality type, we see that the theorem holds for any direct product of a finite number of fundamental IRs. Since any IR of SU_{mn} occurs within some direct product of fundamental IRs,[14] the theorem then follows for any IR of SU_{mn}. We must therefore consider the $(mn-1)$ fundamental IRs $\{1^p\}$, $1 \leq p \leq (mn-1)$ of SU_{mn}. Of these $\{1\}$ is the self representation and the others correspond to the transformations induced by this on the antisymmetric tensors of rank 2, 3, $\cdots$, $(mn-1)$.

Consider first the self representation. Let $\varphi_a (1 \leq a \leq m)$, $\psi_b (1 \leq b \leq n)$ be bases for the self-representations $\{1\}$ of SU_m and SU_n, respectively. If we consider the most general unitary unimodular transformations of the mn product functions $\varphi_a\psi_b$, we clearly have the self-representation of SU_{mn}, i.e., the IR $\{1\}$ of SU_{mn} becomes the IR $\{1\} \otimes \{1\}$ of $SU_m \otimes SU_n$ on restriction from SU_{mn} to $SU_m \otimes SU_n$.

Next we note that for $1 \leq p \leq mn-2$, the product representation $\{1^p\} \otimes \{1\}$ of SU_{mn} contains within it the IR $\{1^{p+1}\}$ of SU_{mn}. This is a simple consequence of the Littlewood method[15] for the reduction of direct products of IRs of special unitary groups. Let us then assume the theorem true for all $\{1^p\}$ with $p \leq p_0$ for some fixed p_0 in the range $1 \leq p_0 \leq mn-2$. Then, using a basic property of plurality type we see that it is true for the product representation $\{1^{p_0}\} \otimes \{1\}$ and, hence, for $\{1^{p_0+1}\}$. Since we have proved the theorem for the IR $\{1\}$, it follows by induction that it is true for all the fundamental IRs of SU_{mn} and, hence, for all IRs of SU_{mn}.

III. ON THE FORMULA FOR THE CHARACTER OF AN IRREDUCIBLE REPRESENTATION OF SU_n

Our starting point is the well-known Weyl formula[12] for the character of an IR $\{l_1, l_2, \cdots l_{n-1}\}$ of SU_n corresponding to an element of SU_n with eigenvalues ϵ_i $(i \leq i \leq n)$ which satisfy $|\epsilon_i| = 1$ and

[12] H. Weyl, *Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1931), p. 381.
[13] M. Hamermesh, *Group Theory* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1962), p. 435.
[14] See G. Racah, "Group Theory and Spectroscopy," Princeton Lectures, 1951 (unpublished), or R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).
[15] D. E. Littlewood, *Theory of Group Characters* (Oxford University Press, London, 1950), 2nd ed., p. 94.

$\epsilon_1\epsilon_2 \cdots \epsilon_n = 1$. The formula in question is

$$\chi(\{l_1, l_2, \cdots l_{n-1}\}, \epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \frac{\begin{vmatrix} \epsilon_1^{l_1+n-1} & \epsilon_1^{l_2+n-2} & \cdots & \epsilon_1^{l_{n-1}+1} & 1 \\ \epsilon_2^{l_1+n-1} & \epsilon_2^{l_2+n-2} & \cdots & \epsilon_2^{l_{n-1}+1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \epsilon_n^{l_1+n-1} & \epsilon_n^{l_2+n-1} & \cdots & \epsilon_n^{l_{n-1}+1} & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon_1^{n-1} & \epsilon_1^{n-2} & \cdots & \epsilon_1 & 1 \\ \epsilon_2^{n-1} & \epsilon_2^{n-2} & \cdots & \epsilon_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \epsilon_n^{n-1} & \epsilon_n^{n-2} & \cdots & \epsilon_n & 1 \end{vmatrix}}. \tag{3.1}$$

It is to be observed that the character χ "belongs to" the symmetric group S_n on the n symbols ϵ_i in the sense that it is invariant under any permutation of them. We shall be able to cast Eq. (3.1) into a form better suited to the work of the next section by introducing certain basic functions which belong to S_n and writing χ in terms of them. The basic functions to which we refer are the so-called[16] elementary symmetric functions and homogeneous product functions. The elementary symmetric functions of the n symbols ϵ_i are

$$\begin{aligned} a_1 &= \sum_i \epsilon_i, \\ a_2 &= \sum_{i<j} \epsilon_i\epsilon_j, \\ a_3 &= \sum_{i<j<k} \epsilon_i\epsilon_j\epsilon_k, \cdots, \\ a_n &= \epsilon_1\epsilon_2\epsilon_3 \cdots \epsilon_n = 1, \end{aligned} \tag{3.2}$$

and the homogeneous product sums are

$$\begin{aligned} h_1 &= \sum_i \epsilon_i = a_1, \\ h_2 &= \sum_{i\leq j} \epsilon_i\epsilon_j, \\ h_3 &= \sum_{i\leq j\leq k} \epsilon_i\epsilon_j\epsilon_k, \cdots. \end{aligned} \tag{3.3}$$

There are only n elementary symmetric functions a_r, i.e., $a_r = 0, r > n$. Also, $a_0 = 1$ and $a_r = 0, r < 0$. For the homogeneous product sums, we also write $h_0 = 1$ and $h_r = 0$, $r < 0$, with the difference that h_r can be defined for any $r > 0$ by Eq. (3.3). However, the h_r for $r > n$ are not algebraically independent of the h_r for $r \leq n$ and are, furthermore, related to the a_r through the equations

$$\begin{aligned} h_1 &= a_1, \\ h_2 &= h_1a_1 - a_1 = a_1^2 - a_2, \\ h_3 &= h_2a_1 - h_1a_2 + a_3 = a_1^3 - 2a_1a_2 + a_3 \cdots, \\ h_r &= h_{r-1}a_1 - h_{r-2}a_2 + \cdots - (-1)^r h_0 a_r. \end{aligned} \tag{3.4}$$

From Eq. (3.4) we can readily derive

$$h_r = \begin{vmatrix} a_1 & a_2 & \cdots & a_{r-1} & a_r \\ 1 & a_1 & \cdots & a_{r-2} & a_{r-1} \\ 0 & 1 & \cdots & & \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & a_1 \end{vmatrix} \tag{3.5}$$

$$= \sum (-1)^{\alpha_2+\alpha_4+\cdots} \frac{(\alpha_1+\alpha_2+\cdots\alpha_n)!}{\alpha_1!\alpha_2!\cdots\alpha_n!} a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n},$$

in which the summation is over all nonnegative α_k satisfying $\sum_k k\alpha_k = r$, a restriction which ensures that each term of the sum be of degree r in the ϵ_i. Equations (3.4) can, of course, equally well be solved for the a_r in terms of the h_r. We then find

$$a_r = \begin{vmatrix} h_1 & h_2 & \cdots & h_{r-1} & h_r \\ 1 & h_1 & \cdots & h_{r-2} & h_{r-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & h_1 \end{vmatrix}, \tag{3.6}$$

which is of the same structure as Eq. (3.5). Since $a_r = 0$ for $r > n$, Eq. (3.6) shows how $h_{n+1}, h_{n+2}, \cdots$ may successively be expressed in terms of the $h_r(r \leq n)$.

It can be proved[16] that the a_r and the h_r with $r \leq n$ provide an algebraic basis for all functions of finite degree in the ϵ_i belonging to S_n. Since χ as given by Eq. (3.1) belongs to S_n and is of finite degree $(l_1 + l_2 + \cdots + l_{n-1})$ in the ϵ_i, this remark should be applicable to it. Indeed χ as given by Eq. (3.1) is exactly the quantity discussed by Littlewood[16] under the na,e "bialternant" and, from his discussion of bialternants, we obtain the result[17]

$$\chi(\{l_1, l_2, \cdots, l_{n-1}\}, \epsilon_1, \cdots, \epsilon_n) = \begin{vmatrix} h_{l_1} & h_{l_1+1} & \cdots & h_{l_1+n-2} \\ h_{l_2-1} & h_{l_2} & \cdots & h_{l_2+n-3} \\ \vdots & & & \\ h_{l_{n-1}-n+2} & & \cdots & h_{l_{n-1}} \end{vmatrix}. \tag{3.7}$$

[16] See D. E. Littlewood, *A University Algebra* (Heinemann Ltd., London, 1958), 2nd ed., Chaps. 5 and 6; or D. E. Littlewood, Ref. 15, Chap. 6.

[17] D. E. Littlewood, Ref. 15, p. 88.

This formula is the one on which our ensuing discussion is principally based. Despite the availability of a formula relating the right sides of Eqs. (3.1) and (3.7) in the work of Littlewood, the result (3.7) does not seem to have been used in previous discussions on the representation theory of unitary groups. Eq. (3.7) is quite distinct from, but presumably equivalent to, the form into which Aghassi and Roman[18] have manipulated the Weyl formula (3.1).

In the case of the IR $\{l, 0, \cdots, 0\}$ or $\{l\}$ simply of SU_n, Eq. (3.7) reduces to

$$\chi(\{l\}, \epsilon_1, \epsilon_2, \cdots \epsilon_n) = h_l, \tag{3.8}$$

since $h_0 = 1$ and $h_r = 0$ for $r < 0$. Hence, Eq. (3.7) expresses the character of the general IR of SU_n as a function of the characters of IRs of SU_n which correspond to Young diagrams of one row or to totally symmetric tensors. This statement is to be contrasted with an analogous but more complicated one made by Aghassi and Roman.[18] In the case of an IR $\{l_1, l_2, \cdots, l_r, 0, \cdots 0\} \equiv \{l_1, \cdots l_r\}$ of SU_n, we see that the determinant on the right side of Eq. (3.7) reduces trivially to one of r rows and columns. In the case of IRs of SU_n corresponding to a Young diagram of a single column with r rows $(0 \leq r \leq n-1)$, we obtain, from Eqs. (3.6) and (3.7), the notable result

$$\chi(\{1^r\}, \epsilon_1, \cdots \epsilon_n) = a_r, \tag{3.9}$$

where we have written $\{1^r\} = \{1, \cdots 1, 0, \cdots 0\}$ with r entries equal to unity on the right. Since the $\{1^r\}$ $(1 \leq r \leq n-1)$ are the fundamental IRs of SU_n, we see that the rth such IR has character a_r. Since any function belonging to S_n, e.g., $\chi(\{l_1, \cdots, l_{n-1}\}, \epsilon_1, \cdots, \epsilon_n)$, can be expressed algebraically in terms of the a_r $(1 \leq r \leq n-1)$, it follows that the character of any IR can be written as a function of the characters of the fundamental IRs. This may be regarded as an expression of the sense in which such representations are fundamental.

In order to give an explicit formula for χ in terms of the a_r, we introduce the Young diagram $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ conjugate to the Young diagram $\{l_1, l_2, \cdots, l_{n-1}\}$. The former is obtained from the latter by transposition, i.e., the numbers $\lambda_1, \lambda_2, \cdots \lambda_p$ refer to the number of boxes in the 1st, 2nd, $\cdots$ pth column of $\{l_1, l_2, \cdots, l_{n-1}\}$. We note that p need not be less than $(n-1)$. Again referring to Littlewood,[19] we obtain the result

$$\chi(\{l_1, l_2, \cdots l_{n-1}\}, \epsilon_1, \cdots \epsilon_n) = \begin{vmatrix} a_{\lambda_1} & a_{\lambda_1+1} & \cdots & a_{\lambda_1+p-1} \\ a_{\lambda_2-1} & a_{\lambda_2} & \cdots & a_{\lambda_2+p-2} \\ \vdots & & & \vdots \\ a_{\lambda_p-p+1} & & \cdots & a_{\lambda_p} \end{vmatrix}. \tag{3.10}$$

Aside from its interest as an explicit statement on the significance of the fundamental IRs of SU_n, this result is of practical importance for the calculation of characters. For example, in the case of IRs of SU_n corresponding to Young diagrams with fewer columns than rows, Eq. (3.10) will involve a determinant of fewer rows and columns than Eq. (3.7). In the particular case of the IR $\{l\}$ of SU_n, the partition conjugate to $\{l\}$ is $\{1^l\}$, and Eq. (3.10) reduces to Eq. (3.8) by virtue of Eq. (3.5). Since $p > n-1$ is possible, Eq. (3.10) can involve a determinant of more than $(n-1)$ rows. The fact that Eq. (3.10) is in this sense more complicated than the form Eq. (3.7) is compensated by the fact that in Eq. (3.10) only a_r with $0 \leq r \leq n$ occur, since $a_r = 0$ for $r > n$. In Eq. (3.7), however, the h_r for $r > n$ are nonvanishing (although algebraically dependent on the h_r for $0 \leq r \leq n$) and can in fact occur.

IV. REDUCTION OF IRs OF SU_{mn} INTO IRs OF $SU_m \otimes SU_n$

We have proved in Sec. II that an IR of SU_{mn} with plurality p_{mn} contains within it only IRs of $SU_m \otimes SU_n$ whose SU_m part has plurality $p_m = p_{mn}$ (mod m) and whose SU_n part has plurality $p_n = p_{mn}$ (mod n). In this section we describe practical methods, based on the character formulas of Sec. III, for actually performing the reduction of IRs of SU_{mn} into IRs of $SU_m \otimes SU_n$. Although general formulas have not been obtained even in the case (Wigner's supermultiplet theory[4]) of SU_4 and $SU_2 \otimes SU_2$, our procedure is quite rapid and economical. We shall illustrate it with reference to the Wigner supermultiplet theory and to its particle physics extension[5] where the reduction in question is from SU_6 to $SU_3 \otimes SU_2$. Of course, application to reductions from SU_9 to $SU_3 \otimes SU_3$,[20] SU_{12} to $SU_3 \otimes SU_4$,[21] etc., can be made along similar lines.

For SU_4 we have IRs $\{l_1, l_2, l_3\}$ corresponding to Young diagrams of at most three rows. The fundamental IRs are three in number $\{1\}$, $\{1, 1\}$, and

[18] J. Aghassi and P. Roman, preprint (1964).
[19] D. E. Littlewood, Ref. 15, p. 89.

[20] H. J. Lipkin, Phys. Letters 9, 203 (1964).
[21] P. G. O. Freund and B. W. Lee, Phys. Rev. Letters 13, 592 (1964).

$\{1, 1, 1\}$ with characters

$$a_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4, \tag{4.1}$$

$$a_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_1\epsilon_4 + \epsilon_2\epsilon_3 + \epsilon_2\epsilon_4 + \epsilon_3\epsilon_4 = a_2^*, \tag{4.2}$$

$$a_3 = \epsilon_2\epsilon_3\epsilon_4 + \epsilon_1\epsilon_3\epsilon_4 + \epsilon_1\epsilon_2\epsilon_4 + \epsilon_1\epsilon_2\epsilon_3 = a_1^*, \tag{4.3}$$

the properties

$$|\epsilon_i| = 1, \quad 1 \le i \le 4, \quad a_4 = \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1, \tag{4.4}$$

of the eigenvalues of a unitary 4×4 matrix having been used. Results (4.1) to (4.3) express the fact that $\{1\}$ is equivalent to the complex conjugate of $\{1, 1, 1\}$, while $\{1, 1\}$ is equivalent to its own complex conjugate. In the Wigner supermultiplet theory, the IR $\{1\}$ operates on the four z-component of spin, z-component of isospin states of a particle with total spin and isospin $\frac{1}{2}$, and is thus a self-representation of arbitrary unitary unimodular transformations of these states. Hence, remembering that the IR $\{l_1, l_2\}$ of SU_2 corresponds to a value $J = \frac{1}{2}(l_1 - l_2)$ of the appropriate total spin, we see that the IR $\{1\}$ of SU_4, when restricted to its spin–isospin $SU_2 \otimes SU_2$ subgroup, yields an IR $\{1\} \otimes \{1\}$ of that subgroup.

To obtain a corresponding result for a general IR of SU_4, we note that the element of SU_4 which describes a general transformation of its $SU_2 \otimes SU_2$ subgroup has the diagonal form

$$\begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{bmatrix} = \begin{bmatrix} \epsilon^{\frac{1}{2}}\eta^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \epsilon^{\frac{1}{2}}\eta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \epsilon^{-\frac{1}{2}}\eta^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \epsilon^{-\frac{1}{2}}\eta^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{bmatrix} \otimes \begin{bmatrix} \eta^{\frac{1}{2}} & 0 \\ 0 & \eta^{-\frac{1}{2}} \end{bmatrix} \tag{4.5}$$

with respect to an ordering (spin up, isospin up), (spin up, isospin down), (spin down, isospin up), (spin down, isospin down) of the basis states of the self-representation, The parameters ϵ and η refer, respectively, to the spin and isospin parts of the product transformation. From (4.5) we see easily[22] that

$$a_1 = (\epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}})(\eta^{\frac{1}{2}} + \eta^{-\frac{1}{2}}), \tag{4.6}$$

so that on restriction from SU_4 to $SU_2 \otimes SU_2$,

$$\chi(\{1\}) = a_1 \to \chi_s(\{1\})\chi_i(\{1\}), \tag{4.7}$$

where we have suppressed the ϵ, η arguments and the subscripts s and i to refer to the spin and isospin SU_2 groups. Thus the IR $\{1\}$ of SU_4 yields an IR $\{1\} \otimes \{1\}$ of $SU_2 \otimes SU_2$, a decomposition which clearly applies to the representation $\{1, 1, 1\}$ as well. From (4.5) we also get

$$a_2 = \epsilon + 1 + \epsilon^{-1} + \eta + 1 + \eta^{-1}, \tag{4.8}$$

so that[22]

$$\chi(\{1, 1\}) = a_2 \to \chi_s(\{2\})\chi_i(\{0\}) + \chi_s(\{0\})\chi_i(\{2\}), \tag{4.9}$$

i.e., the IR $\{1, 1\}$ of SU_4 yields the IRs $\{2\} \otimes \{0\}$ and $\{0\} \otimes \{2\}$ of $SU_2 \otimes SU_2$.

It now follows that the splitting of the general IR $\{l_1, l_2, l_3\}$ into IRs of $SU_2 \otimes SU_2$ can be obtained directly from Eq. (3.10) by systematic use of the formula

$$\chi_{s,i}(\{k\})\chi_{s,i}(\{k'\}) = \chi_{s,i}(\{k + k'\}) + \chi_{s,i}(\{k + k' - 2\}) + \cdots + \chi_{s,i}(\{|k - k'|\}), \tag{4.10}$$

corresponding to the well known Clebsch–Gordan series for SU_2. For IRs of SU_4 with fewer columns than rows, this is the best way to proceed. However, for other IRs of SU_4. it is much easier to use Eq. (3.7), since it is possible to give a general result for the reduction for an IR $\{l\}$ of SU_4 into a sum of IRs of $SU_2 \otimes SU_2$. This leads, of course, to the well known result[13] that on restriction from SU_4 to $SU_2 \otimes SU_2$, we get for the IR $\{l\}$ of SU_4 the reduction

$$\{l\} \to \{l\} \otimes \{l\} + \{l - 2\} \otimes \{l - 2\} + \{l - 4\} \otimes \{l - 4\} \cdots, \tag{4.11}$$

terminating with $\{1\} \otimes \{1\}$ for l odd and $\{0\} \otimes \{0\}$ for l even. We could alternatively say that when restriction to $SU_2 \otimes SU_2$ is made, the IR $\{l\}$ of SU_4 yields once and only once each IR of $SU_2 \otimes SU_2$ which has both of its parts corresponding to the same Young diagram with l boxes and not more than two rows. The statement, of course, reduces to the previous one since the IRs $\{l + e, e\}$ and $\{l\}$ of SU_2 are equivalent for e integral; however, it resembles more closely the corresponding state-

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[22] The $IR\{l_1, l_2\}$ of SU_2 has total spin value $J = \frac{1}{2}(l_1 - l_2)$ and, for an element of SU_2 with diagonal form

$$\begin{pmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{pmatrix},$$

has character [cf. Eq. (3.1)] $\sum^J{}_{i=-J}\epsilon^i$. For $J = \frac{1}{2}$ this sum is simply $(\epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}})$.

ment regarding the reduction of a totally symmetric IR[23] of SU_{mn} with respect to $SU_m \otimes SU_n$.

To obtain our result we prove the corresponding character relationship, i.e.,

$$h_l = \chi_s(\{l\})\chi_i(\{l\}) + \chi_s(\{l-2\})\chi_i(\{l-2\}) + \cdots . \tag{4.12}$$

Clearly (4.12) is true for $l = 0$ and $l = 1$ using $h_1 = a_1$ and Eq. (4.6). If we assume it true for all l up to some fixed l_0 and on this basis prove it true for $(l_0 + 1)$, then it follows for all l by induction. If Eq. (4.12) is true for all $l \leq l_0$, then

$$h_l - h_{l-2} = \chi_s(\{l\})\chi_i(\{l\}) \tag{4.13}$$

follows for all $l \leq l_0$ also. Hence, using the last line of Eq. (3.4), and Eq. (4.11) for $l = l_0, l_0 - 1, l_0 - 2, l_0 - 3$, we get

$$\begin{aligned} h_{l_0+1} - h_{l_0-1} = a_1\chi_s(\{l_0\})\chi_i(\{l_0\}) &- a_2\chi_s(\{l_0-1\})\chi_i(\{l_0-1\}) \\ &+ a_3\chi_s(\{l_0-2\})\chi_i(\{l_0-2\}) \\ &- a_4\chi_s(\{l_0-3\})\chi_i(\{l_0-3\}), \end{aligned} \tag{4.14}$$

which, upon using Eqs. (4.7), (4.8), (4.3), and (4.4), together with Eq. (4.10), reduces to

$$h_{l_0+1} - h_{l_0-1} = \chi_s(\{l_0+1\})\chi_i(\{l_0+1\}). \tag{4.15}$$

By assumption Eq. (4.12) holds for $l \leq l_0$, and since we have proved it true on this basis for $l = l_0 + 1$, its validity for all l follows by induction. Now with the aid of Eqs. (4.14) and (4.10), we can directly reduce

$$\chi(\{l_1, l_2, l_3\}) = \begin{vmatrix} h_{l_1} & h_{l_1+1} & h_{l_1+2} \\ h_{l_2-1} & h_{l_2} & h_{l_2+1} \\ h_{l_3-2} & h_{l_3-1} & h_{l_3} \end{vmatrix} \tag{4.16}$$

to a sum of products $\chi_s\chi_i$. We do not in this case reproduce the more specific results already available in the literature[13] since our primary purpose here has been to illustrate our general approach. It is to be noted that this approach contrasts with previous ones in that it is directly applicable rather than being recursive.

We now go on to the case of SU_6 and $SU_2 \otimes SU_3$. For SU_6 we have five fundamental IRs with characters $a_r = (\{1^r\})$, $1 \leq r \leq 5$, given explicitly by

$$a_1 = a_5^* = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6, \tag{4.17}$$

[23] M. Moshinsky, J. Math. Phys. 4, 1128 (1963). See also T. A. Brody, M. Moshinsky, and I. Renero, *Recursion Relations for the Wigner Coefficients of Unitary Groups*, preprint (1964).

$$\begin{aligned} a_2 = a_4^* = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_1\epsilon_4 + \epsilon_1\epsilon_5 + \epsilon_1\epsilon_6 &+ \epsilon_2\epsilon_3 + \epsilon_2\epsilon_4 + \epsilon_2\epsilon_5 + \epsilon_2\epsilon_6 + \epsilon_3\epsilon_4 \\ &+ \epsilon_3\epsilon_5 + \epsilon_3\epsilon_6 + \epsilon_4\epsilon_5 + \epsilon_4\epsilon_6 + \epsilon_5\epsilon_6, \end{aligned} \tag{4.18}$$

$$a_3 = a_3^* = \epsilon_1\epsilon_2\epsilon_3 + \epsilon_1\epsilon_2\epsilon_4 + \cdots \epsilon_4\epsilon_5\epsilon_6, \tag{4.19}$$

where we have used $a_6 = \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\epsilon_6 = 1$ and $|\epsilon_i| = 1$, $1 \leq i \leq 6$. The self representation $\{1\}$ of SU_6 operates on the six states of a spin-$\frac{1}{2}$ quark, a quark[11] being associated with the IR $\{1\}$ of SU_3 whose three basis states are characterized by $I = \frac{1}{2}$, $I_z = \pm\frac{1}{2}$, $Y = \frac{1}{3}$, and $I = 0$, $Y = -\frac{2}{3}$. In other words when restriction from SU_6 to $SU_2 \otimes SU_3$ is made, we get the result

$$\{1\} \to \{1\} \otimes \{1\}. \tag{4.20}$$

Here, as always in what follows, the first symbol refers to the SU_2 part of the product IR. Thus to perform the reduction of SU_6 to $SU_2 \otimes SU_3$, we consider elements of SU_6 with diagonal form

$$\begin{aligned} &\begin{bmatrix} \epsilon_1 &&&&& \\ & \epsilon_2 &&&& \\ && \epsilon_3 &&& \\ &&& \epsilon_4 && \\ &&&& \epsilon_5 & \\ &&&&& \epsilon_6 \end{bmatrix} \\ &= \begin{bmatrix} \epsilon^{\frac{1}{2}}\eta_1 &&&&& \\ & \epsilon^{\frac{1}{2}}\eta_2 &&&& \\ && \epsilon^{\frac{1}{2}}\eta_3 &&& \\ &&& \epsilon^{-\frac{1}{2}}\eta_1 && \\ &&&& \epsilon^{-\frac{1}{2}}\eta_2 & \\ &&&&& \epsilon^{-\frac{1}{2}}\eta_3 \end{bmatrix} \\ &= \begin{bmatrix} \epsilon^{\frac{1}{2}} & 0 \\ 0 & \epsilon^{-\frac{1}{2}} \end{bmatrix} \otimes \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix}, \end{aligned} \tag{4.21}$$

where ϵ refers to the spin-SU_2 subgroup of SU_6, and the η_i satisfy $\eta_1\eta_2\eta_3 = 1$ and refer to the SU_3 subgroup.

From Eqs. (4.17) to (4.19) and (4.21) one easily obtains

$$a_1 = \chi_s(\{1\})\chi_f(\{1\}), \tag{4.22}$$

$$a_2 = \chi_s(\{0\})\chi_f(\{2\}) + \chi_s(\{2\})\chi_f(\{1, 1\}), \tag{4.23}$$

$$a_3 = \chi_s(\{1\})\chi_f(\{2, 1\}) + \chi_s(\{3\})\chi_f(\{0\}), \tag{4.24}$$

$$a_4 = \chi_s(\{0\})\chi_f(\{2, 2\}) + \chi_s(\{2\})\chi_f(\{1\}), \tag{4.25}$$

$$a_5 = \chi_s(\{1\})\chi_f(\{1, 1\}), \tag{4.26}$$

so that on restriction from SU_6 to $SU_2 \otimes SU_3$

$$\{1\} \to \{1\} \otimes \{1\}, \tag{4.27}$$

TABLE I. Irreducible representations of SU_6 and their reduction with respect to $SU_2 \otimes SU_3$. For each IR of SU_6 both the Young diagram and highest weight designations are given. The $SU_2 \otimes SU_3$ content of each IR of SU_6 is given first in Young-diagram notation and secondly in a hybrid notation which refers to SU_2 parts by their total spin values and to SU_3 parts by highest weight notation.

IR of SU_6	Dimension	Hexality	$SU_2 \times SU_3$ content	Duality of SU_2 part	Triality of SU_3 part
$\{1\}$ $(1, 0, 0, 0, 0)$	6	1	$\{1\} \otimes \{1\}$ $\frac{1}{2} \otimes (1, 0)$	1	1
$\{1, 1\}$ $(0, 1, 0, 0, 0)$	15	2	$\{0\} \otimes \{2\} + \{2\} \otimes \{1, 1\}$ $0 \otimes (2, 0) + 1 \otimes (0, 1)$	0	2
$\{1, 1, 1\}$ $(0, 0, 1, 0, 0)$	20	3	$\{1\} \otimes \{2, 1\} + \{3\} \otimes \{0\}$ $\frac{1}{2} \otimes (1, 1) + 3/2 \otimes (0, 0)$	1	0
$\{1, 1, 1, 1\}$ $(0, 0, 0, 1, 0)$	15	4	$\{0\} \otimes \{2, 2\} + \{2\} \otimes \{1\}$ $0 \otimes (0, 2) + 1 \otimes (1, 0)$	0	1
$\{1, 1, 1, 1, 1\}$ $(0, 0, 0, 0, 1)$	6	5	$\{1\} \otimes \{1, 1\}$ $\frac{1}{2} \otimes (0, 1)$	1	2
$\{2\}$ $(2, 0, 0, 0, 0)$	21	2	$\{2\} \otimes \{2\} + \{0\} \otimes \{1, 1\}$ $1 \otimes (2, 0) + 0 \otimes (0, 1)$	0	2
$\{2, 1, 1, 1, 1\}$ $(1, 0, 0, 0, 1)$	35	0	$\{2\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\} + \{2\} \otimes \{0\}$ $1 \otimes (1, 1) + 0 \otimes (1, 1) + 1 \otimes (0, 0)$	0	0
$\{3\}$ $(3, 0, 0, 0, 0)$	56	3	$\{3\} \otimes \{3\} + \{1\} \otimes \{2, 1\}$ $3/2 \otimes (3, 0) + \frac{1}{2} \otimes (1, 1)$	1	0
$\{2, 1\}$ $(1, 1, 0, 0, 0)$	70	3	$\{1\} \otimes \{3\} + \{3\} \otimes \{2, 1\}$ $\{1\} \otimes \{2, 1\} + \{1\} \otimes \{0\}$ $\frac{1}{2} \otimes (3, 0) + 3/2 \otimes (1, 1)$ $\frac{1}{2} \otimes (1, 1) + \frac{1}{2} \otimes (0, 0)$	1	0
$\{2, 1, 1, 1\}$ $(1, 0, 0, 1, 0)$	84	5	$\{1\} \otimes \{3, 2\} + \{1\} \otimes \{1, 1\} + \{3\}$ $\otimes \{2\} + \{1\} \otimes \{2\} + \{3\} \otimes \{1, 1\}$ $\frac{1}{2} \otimes (1, 2) + \frac{1}{2} \otimes (0, 1) + 3/2 \otimes (2, 0)$ $+ \frac{1}{2} \otimes (2, 0) + 3/2 \otimes (0, 1)$	1	2
$\{2, 2\}$ $(0, 2, 0, 0, 0)$	105	4	$\{4\} \otimes \{2, 2\} + \{2\} \otimes \{1\} + \{2\} \otimes \{3, 1\} + \{0\} \otimes \{4\} +$ $\{0\} \otimes \{2, 2\}$ $2 \otimes (0, 2) + 1 \otimes (1, 0) + 1 \otimes (2, 1) + 0 \otimes (4, 0) + 0 \otimes (0, 2)$	0	1
$\{2, 1, 1\}$ $(1, 0, 1, 0, 0)$	105	4	$\{2\} \otimes \{3, 1\} + \{2\} \otimes \{2, 2\} + \{2\} \otimes \{1\} + \{0\} \otimes$ $\{3, 1\} + \{0\} \otimes \{1\} + \{4\} \otimes \{1\}$ $1 \otimes (2, 1) + 1 \otimes (0, 2) + 1 \otimes (1, 0)$ $0 \otimes (2, 1) + 0 \otimes (1, 0) + 2 \otimes (1, 0)$	0	1
$\{4\}$ $(4, 0, 0, 0, 0)$	126	4	$\{4\} \otimes \{3, 1\} + \{2\} \otimes \{4, 0\} + \{0\} \otimes \{2, 2\}$ $2 \otimes (2, 1) + 1 \otimes (4, 0) + 0 \otimes (0, 2)$	0	1
$\{2, 2, 2\}$ $(0, 0, 2, 0, 0)$	175	0	$\{2\} \otimes \{4, 2\} + \{2\} \otimes \{0\} + \{0\} \otimes \{3\} + \{0\} \otimes \{3, 3\} +$ $\{6\} \otimes \{0\} + \{4\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\}$ $1 \otimes (2, 2) + 1 \otimes (0, 0) + 0 \otimes (3, 0) + 0 \otimes (0, 3) +$ $3 \otimes (0, 0) + 2 \otimes (1, 1) + 1 \otimes (1, 1)$	0	0
$\{2, 2, 1, 1\}$ $(0, 1, 0, 1, 0)$	189	0	$\{0\} \otimes \{4, 2\} + \{0\} \otimes \{2, 1\} + \{0\} \otimes \{0\} + \{2\} \otimes$ $\{2, 1\} + \{2\} \otimes \{3\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{3, 3\} +$ $\{4\} \otimes \{2, 1\} + \{4\} \otimes \{0\}$ $0 \otimes (2, 2) + 0 \otimes (1, 1) + 0 \otimes (0, 0) + 1 \otimes (1, 1) + 1 \otimes$ $(3, 0) + 1 \otimes (1, 1) + 1 \otimes (0, 3) + 2 \otimes (1, 1) + 2 \otimes (0, 0)$	0	0
$\{3, 1\}$ $(2, 1, 0, 0, 0)$	210	4	$\{4\} \otimes \{4\} + \{2\} \otimes \{3, 1\} + \{2\} \otimes \{2, 2\} + \{2\} \otimes \{3, 1\} +$ $\{2\} \otimes \{1\} + \{0\} \otimes \{3, 1\} + \{0\} \otimes \{1\}$ $2 \otimes (4, 0) + 1 \otimes (2, 1) + 1 \otimes (0, 2) + 1 \otimes (2, 1) + 1 \otimes$ $(1, 0) + 0 \otimes (2, 1) + 0 \otimes (1, 0)$	0	1
$\{2, 2, 1\}$ $(0, 1, 1, 0, 0)$	210	5	$\{5\} \otimes \{1, 1\} + \{3\} \otimes \{1, 1\} + \{3\} \otimes \{2\} + \{3\} \otimes$ $\{3, 2\} + \{1\} \otimes \{3, 2\} + \{1\} \otimes \{2\} + \{1\} \otimes \{1, 1\} +$ $\{1\} \otimes \{4, 1\}$ $5/2 \otimes (0, 1) + 3/2 \otimes (0, 1) + 3/2 \otimes (2, 0) + 3/2 \otimes$ $(1, 2) + \frac{1}{2} \otimes (1, 2) + \frac{1}{2} \otimes (2, 0) + \frac{1}{2} \otimes (0, 1) + \frac{1}{2} \otimes (3, 1)$	1	2

TABLE I. (*Continued*).

IR of SU_6	Dimension	Hexality	$SU_2 \times SU_3$ content	Duality of SU_2 part	Triality of SU_3 part
{5} (5, 0, 0, 0, 0)	252	5	$\{5\} \otimes \{5\} + \{3\} \otimes \{4, 1\} + \{1\} \otimes \{3, 2\}$ $\frac{5}{2} \otimes (5, 0) + \frac{3}{2} \otimes (3, 1) + \frac{1}{2} \otimes (1, 2)$	1	2
{3, 1, 1, 1} (2, 0, 0, 1, 0)	280	0	$\{4\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\} + \{4\} \otimes \{3, 3\} + \{2\} \otimes \{3, 3\} + \{0\} \otimes \{3, 3\} + \{0\} \otimes \{3\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{4, 2\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{0\}$ $2 \otimes (1, 1) + 0 \otimes (1, 1) + 2 \otimes (0, 3) + 1 \otimes (3, 0) + 0 \otimes (0, 3) + 0 \otimes (3, 0) + 1 \otimes (1, 1) + 1 \otimes (2, 2) + 1 \otimes (1, 1) + 1 \otimes (0, 0)$	0	0
{4, 2, 2, 2, 2} (2, 0, 0, 0, 2)	405	0	$\{4\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{0\} \otimes \{4, 2\} + \{2\} \otimes \{3\} + \{2\} \otimes \{3, 3\} + \{4\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\} + \{4\} \otimes \{0\} + \{0\} \otimes \{0\}$ $2 \otimes (2, 2) + 1 \otimes (2, 2) + 0 \otimes (2, 2) + 1 \otimes (3, 0) + 1 \otimes (0, 3) + 2 \otimes (1, 1) + 1 \otimes (1, 1) + 1 \otimes (1, 1) + 0 \otimes (1, 1) + 2 \otimes (0, 0) + 0 \otimes (0, 0)$	0	0
{6} (6, 0, 0, 0, 0)	462	0	$\{6\} \otimes \{6\} + \{4\} \otimes \{5, 1\} + \{2\} \otimes \{4, 2\} + \{0\} \otimes \{3, 3\}$ $3 \otimes (6, 0) + 2 \otimes (4, 1) + 1 \otimes (2, 2) + 0 \otimes (0, 3)$	0	0
{3, 3} (0, 3, 0, 0, 0)	490	0	$\{0\} \otimes \{6\} + \{0\} \otimes \{4, 2\} + \{0\} \otimes \{0\} + \{2\} \otimes \{5, 1\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{3, 0\} + \{2\} \otimes \{3, 3\} + \{4\} \otimes \{4, 2\} + \{4\} \otimes \{2, 1\} + \{6\} \otimes \{3, 3\}$ $0 \otimes (6, 0) + 0 \otimes (2, 2) + 0 \otimes (0, 0) + 1 \otimes (4, 1) + 1 \otimes (1, 1) + 1 \otimes (3, 0) + 1 \otimes (0, 3) + 2 \otimes (2, 2) + 2 \otimes (1, 1) + 3 \otimes (0, 3)$	0	0
{3, 2, 2, 1, 1} (1, 0, 1, 0, 1)	540	3	$\{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{3\} \otimes \{3\} + \{1\} \otimes \{3\} + \{3\} \otimes \{3, 3\} + \{1\} \otimes \{3, 3\} + \{5\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{5\} \otimes \{0\} + \{3\} \otimes \{0\} + \{1\} \otimes \{0\}$ $\frac{3}{2} \otimes (2, 2) + \frac{1}{2} \otimes (2, 2) + \frac{1}{2} \otimes (2, 2) + \frac{3}{2} \otimes (3, 0) + \frac{1}{2} \otimes (3, 0) + \frac{3}{2} \otimes (0, 3) + \frac{1}{2} \otimes (0, 3) + \frac{5}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{5}{2} \otimes (0, 0) + \frac{3}{2} \otimes (0, 0) + \frac{1}{2} \otimes (0, 0)$	1	0
{3, 2, 2, 2} (1, 0, 0, 2, 0)	560	3	$\{1\} \otimes \{5, 4\} + \{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{3\} \otimes \{3, 3\} + \{1\} \otimes \{3, 3\} + \{5\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{5\} \otimes \{3\} + \{3\} \otimes \{3\} + \{1\} \otimes \{3\} + \{3\} \otimes \{0\}$ $\frac{1}{2} \otimes (1, 4) + \frac{3}{2} \otimes (2, 2) + \frac{1}{2} \otimes (2, 2) + \frac{3}{2} \otimes (0,3) + \frac{1}{2} \otimes (0, 3) + \frac{5}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{5}{2} \otimes (3, 0) + \frac{3}{2} \otimes (3, 0) + \frac{1}{2} \otimes (3, 0) + \frac{3}{2} \otimes (0, 0)$	1	0
{5, 1, 1, 1, 1} (4, 0, 0, 0, 1)	700	3	$\{5\} \otimes \{5, 1\} + \{3\} \otimes \{5, 1\} + \{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{3\} \otimes \{3\} + \{1\} \otimes \{3\} + \{1\} \otimes \{3, 3\} + \{5\} \otimes \{3\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\}$ $5/2 \otimes (4, 1) + 3/2 \otimes (4, 1) + 3/2 \otimes (2, 2) + \frac{1}{2} \otimes (2, 2) + 3/2 \otimes (3, 0) + \frac{1}{2} \otimes (3, 0) + \frac{1}{2} \otimes (0, 3) + 5/2 \otimes (3, 0) + 3/2 \otimes (1, 1) + \frac{1}{2} \otimes (1, 1)$	1	0
{3, 3, 3} (0, 0, 3, 0, 0)	980	3	$\{3\} \otimes \{6, 3\} + \{1\} \otimes \{5, 1\} + \{1\} \otimes \{5, 4\} + \{5\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{3\} \otimes \{3\} + \{3\} \otimes \{3, 3\} + \{7\} \otimes \{2, 1\} + \{5\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{9\} \otimes \{0\} + \{5\} \otimes \{0\} + \{3\} \otimes \{0\}$ $\frac{3}{2} \otimes (3, 3) + \frac{1}{2} \otimes (4, 1) + \frac{1}{2} \otimes (1, 4) + \frac{5}{2} \otimes (2, 2) + \frac{3}{2} \otimes (2, 2) + \frac{1}{2} \otimes (2, 2) + \frac{3}{2} \otimes (3, 0) + \frac{3}{2} \otimes (0, 3) + \frac{7}{2} \otimes (1, 1) + \frac{5}{2} \otimes (1, 1) + \frac{3}{2} \otimes (1, 1) + \frac{1}{2} \otimes (1, 1) + \frac{9}{2} \otimes (0, 0) + \frac{5}{2} \otimes (0, 0) + \frac{3}{2} \otimes (0, 0)$	1	0
{5, 1}	1050	0	$\{4\} \otimes \{6\} + \{6\} \otimes \{5, 1\} + \{4\} \otimes \{5, 1\} + \{2\} \otimes \{5, 1\} + \{4\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{0\} \otimes \{4, 2\} + \{2\} \otimes \{3, 3\} + \{4\} \otimes \{3\} + \{2\} \otimes \{3\} + \{2\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\}$	0	0

TABLE I. (*Continued*).

IR of SU_6	Dimension	Hexality	$SU_2 \times SU_3$ content	Duality of SU_2 part	Triality of SU_3 part
(4, 1, 0, 0, 0)			$2 \otimes (6, 0) + 3 \otimes (4, 1) + 2 \otimes (4, 1) + 1 \otimes (4, 1) + 2 \otimes (2, 2) + 1 \otimes (2, 2) + 0 \otimes (2, 2) + 1 \otimes (0, 3) + 2 \otimes (3, 0) + 1 \otimes (3, 0) + 1 \otimes (1, 1) + 0 \otimes (1, 1)$		
{4, 2}	1134	0	$\{2\} \otimes \{6\} + \{4\} \otimes \{5, 1\} + \{2\} \otimes \{5, 1\} + \{0\} \otimes \{5, 1\} + \{6\} \otimes \{4, 2\} + \{4\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{4\} \otimes \{3, 3\} + \{0\} \otimes \{3, 3\} + \{4\} \otimes \{3\} + \{2\} \otimes \{3\} + \{0\} \otimes \{3\} + \{4\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\} + \{2\} \otimes \{0\}$	0	0
(2, 2, 0, 0, 0)			$1 \otimes (6, 0) + 2 \otimes (4, 1) + 1 \otimes (4, 1) + 0 \otimes (4, 1) + 3 \otimes (2, 2) + 2 \otimes (2, 2) + 1 \otimes (2, 2) + 1 \otimes (2, 2) + 2 \otimes (0, 3) + 0 \otimes (0, 3) + 2 \otimes (3, 0) + 1 \otimes (3, 0) + 0 \otimes (3, 0) + 2 \otimes (1, 1) + 1 \otimes (1, 1) + 1 \otimes (1, 1) + 0 \otimes (1, 1) + 1 \otimes (0, 0)$		
{4, 2, 1, 1, 1}	1134	3	$\{3\} \otimes \{5, 1\} + \{1\} \otimes \{5, 1\} + \{5\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{5\} \otimes \{3, 1\} + \{3\} \otimes \{3\} + \{3\} \otimes \{3\} + \{1\} \otimes \{3\} + \{1\} \otimes \{3\} + \{3\} \otimes \{3, 3\} + \{1\} \otimes \{3, 3\} + \{5\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{3\} \otimes \{0\} + \{1\} \otimes \{0\}$	1	0
(2, 1, 0, 0, 1)			$3/2 \otimes (4, 1) + \tfrac{1}{2} \otimes (4, 1) + 5/2 \otimes (2, 2) + 3/2 \otimes (2, 2) + 3/2 \otimes (2, 2) + \tfrac{1}{2} \otimes (2, 2) + \tfrac{1}{2} \otimes (2, 2) + 5/2 \otimes (3, 0) + 3/2 \otimes (3, 0) + 3/2 \otimes (3, 0) + \tfrac{1}{2} \otimes (3, 0) + \tfrac{1}{2} \otimes (3, 0) + 3/2 \otimes (0, 3) + \tfrac{1}{2} \otimes (0, 3) + 5/2 \otimes (1, 1) + 3/2 \otimes (1, 1) + 3/2 \otimes (1, 1) + 3/2 \otimes (1, 1) + \tfrac{1}{2} \otimes (1, 1) + \tfrac{1}{2} \otimes (1, 1) + \tfrac{1}{2} \otimes (1, 1) + 3/2 \otimes (0, 0) + \tfrac{1}{2} \otimes (0, 0)$		
{3, 3, 2, 1}	1960	3	$\{1\} \otimes \{6, 3\} + \{3\} \otimes \{5, 1\} + \{1\} \otimes \{5, 1\} + \{3\} \otimes \{5, 4\} + \{1\} \otimes \{5, 4\} + \{5\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{3\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{1\} \otimes \{4, 2\} + \{5\} \otimes \{3\} + \{3\} \otimes \{3\} + \{3\} \otimes \{3\} + \{1\} \otimes \{3\} + \{1\} \otimes \{3\} + \{5\} \otimes \{3, 3\} + \{3\} \otimes \{3, 3\} + \{3\} \otimes \{3, 3\} + \{1\} \otimes \{3, 3\} + \{1\} \otimes \{3, 3\} + \{7\} \otimes \{2, 1\} + \{5\} \otimes \{2, 1\} + \{5\} \otimes \{2, 1\} + \{5\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{7\} \otimes \{0\} + \{5\} \otimes \{0\} + \{3\} \otimes \{0\} + \{1\} \otimes \{0\}$	1	0
(0, 1, 1, 1, 0)			$\tfrac{1}{2} \otimes (3, 3) + \tfrac{3}{2} \otimes (4, 1) + \tfrac{1}{2} \otimes (4, 1) + \tfrac{3}{2} \otimes (1, 4) + \tfrac{1}{2} \otimes (1, 4) + \tfrac{5}{2} \otimes (2, 2) + \tfrac{3}{2} \otimes (2, 2) + \tfrac{3}{2} \otimes (2, 2) + \tfrac{3}{2} \otimes (2, 2) + \tfrac{1}{2} \otimes (2, 2) + \tfrac{1}{2} \otimes (2, 2) + \tfrac{1}{2} \otimes (2, 2) + \tfrac{5}{2} \otimes (3, 0) + \tfrac{3}{2} \otimes (3, 0) + \tfrac{3}{2} \otimes (3, 0) + \tfrac{1}{2} \otimes (3, 0) + \tfrac{1}{2} \otimes (3, 0) + \tfrac{5}{2} \otimes (0, 3) + \tfrac{3}{2} \otimes (0, 3) + \tfrac{3}{2} \otimes (0, 3) + \tfrac{1}{2} \otimes (0, 3) + \tfrac{1}{2} \otimes (0, 3) + \tfrac{7}{2} \otimes (1, 1) + \tfrac{5}{2} \otimes (1, 1) + \tfrac{5}{2} \otimes (1, 1) + \tfrac{5}{2} \otimes (1, 1) + \tfrac{3}{2} \otimes (1, 1) + \tfrac{3}{2} \otimes (1, 1) + \tfrac{3}{2} \otimes (1, 1) + \tfrac{3}{2} \otimes (1, 1) + \tfrac{1}{2} \otimes (1, 1) + \tfrac{1}{2} \otimes (1, 1) + \tfrac{1}{2} \otimes 1, 1) + \tfrac{7}{2} \otimes (0, 0) + \tfrac{5}{2} \otimes (0, 0) + \tfrac{3}{2} \otimes (0, 0) + \tfrac{1}{2} \otimes (0, 0)$		
{6, 3, 3, 3, 3}	2695	0	$\{6\} \otimes \{6, 3\} + \{4\} \otimes \{6, 3\} + \{2\} \otimes \{6, 3\} + \{0\} \otimes \{6, 3\} + \{4\} \otimes \{5, 1\} + \{2\} \otimes \{5, 1\} + \{4\} \otimes \{5, 4\} + \{2\} \otimes \{5, 4\} + \{6\} \otimes \{4, 2\} + \{4\} \otimes \{4, 2\} + \{4\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{2\} \otimes \{4, 2\} + \{0\} \otimes \{4, 2\} + \{4\} \otimes \{3\} + \{2\} \otimes \{3\} + \{0\} \otimes \{3\} + \{4\} \otimes \{3, 3\} + \{2\} \otimes \{3, 3\} + \{0\} \otimes \{3, 3\} + \{6\} \otimes \{2, 1\} + \{4\} \otimes \{2, 1\} + \{4\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{2\} \otimes \{2, 1\} + \{0\} \otimes \{2, 1\} + \{6\} \otimes \{0\} + \{2\} \otimes \{0\}$	0	0
(3, 0, 0, 0, 3)			$3 \otimes (3, 3) + 2 \otimes (3, 3) + 1 \otimes (3, 3) + 0 \otimes (3, 3) + 2 \otimes (4, 1) + 1 \otimes (4, 1) + 2 \otimes (1, 4) + 1 \otimes (1, 4) + 3 \otimes (2, 2) + 2 \otimes (2, 2) + 2 \otimes (2, 2) + 1 \otimes (2, 2) + 1 \otimes (2, 2) + 1 \otimes (2, 2) + 0 \otimes (2, 2) + 2 \otimes (3, 0) + 1 \otimes (3, 0) + 0 \otimes (3, 0) + 2 \otimes (0, 3) + 1 \otimes (0, 3) + 0 \otimes (0, 3) + 3 \otimes (1, 1) + 2 \otimes (1, 1) + 2 \otimes (1, 1) + 1 \otimes (1, 1) + 1 \otimes (1, 1) + 0 \otimes (1, 1) + 3 \otimes (0, 0) + 1 \otimes (0, 0)$		

$$\{1^2\} \to \{0\} \otimes \{2\} + \{2\} \otimes \{1^2\}, \tag{4.28}$$

$$\{1^3\} \to \{1\} \otimes \{2, 1\} + \{3\} \otimes \{0\}, \tag{4.29}$$

$$\{1^4\} \to \{0\} \otimes \{2^2\} + \{2\} \otimes \{1\}, \tag{4.30}$$

$$\{1^5\} \to \{1\} \otimes \{1^2\}. \tag{4.31}$$

It now follows on using Eqs. (4.22) to (4.26), Eq. (4.10), and the known[24] Clebsch–Gordan series for SU_3, that we can directly obtain from Eq. (3.10) the reduction of any IR of SU_6 with respect to $SU_2 \otimes SU_3$. In the present case it is important in terms of algebraic effort to use Eq. (3.10) when the IR of SU_6 in question has fewer columns than rows and otherwise to use Eq. (3.7). Again the advantage of using Eq. (3.7) stems from the fact that, on restriction from SU_6 to $SU_2 \otimes SU_3$, we get

$$h_l = \chi_s(\{l\})\chi_f(\{l\}) + \chi_s(\{l-2\})\chi_f(\{l-1, 1\}) + \chi_s(\{l-4\})\chi_f(\{l-2, 2\}) + \cdots, \tag{4.32}$$

the last term being $\chi_s(\{1\})\chi_f(\{\frac{1}{2}l, \frac{1}{2}l\})$ for l even and $\chi_s(\{1\})\chi_f(\{\frac{1}{2}(l+1), \frac{1}{2}(l-1)\})$ for l odd. This corresponds to saying that when restriction to $SU_2 \otimes SU_3$ is made, the IR $\{l\}$ of SU_6 yields once and only once each IR of $SU_2 \otimes SU_3$ which has both of its parts corresponding to the same Young diagram with l boxes and not more than two rows. The proof proceeds along the same lines as for SU_4.

To illustrate the ready applicability of the present methods we consider the IRs of SU_6 of 56, 35, and 70 dimensions which have been employed in the currently popular SU_6 theory. The IR with 56 dimensions is $\{3\}$, and from (4.32)

$$\{3\} \to \{3\} \otimes \{3\} + \{1\} \otimes \{2, 1\}; \tag{4.33}$$

"56" → "spin-$\frac{3}{2}$ decuplet" + "spin-$\frac{1}{2}$ octet."

The IR with 35 dimensions is the regular representation $\{2, 1, 1, 1, 1\}$. The shape conjugate to $\{2, 1, 1, 1, 1\}$ is $\{5, 1\}$. Hence, using Eq. (3.10), and then Eqs. (4.22) and (4.26), we get

$$\chi(\{2,1,1,1,1\}) = \begin{vmatrix} a_5 & a_6 \\ 1 & a_1 \end{vmatrix} \tag{4.34}$$

$$= \chi_s(\{1\})\chi_f(\{1,1\})\chi_s(\{1\})\chi_f(\{1\}) - \chi_s(\{0\})\chi_f(\{0\}). \tag{4.35}$$

Now using the Clebsch–Gordan series for SU_2 and SU_3 the result

$$\{2, 1, 1, 1, 1\} \to \{2\} \otimes \{2, 1\} + \{2\} \otimes \{0\} + \{0\} \otimes \{2, 1\}; \tag{4.36}$$

"35" → "spin-1 octet" + "spin-1 singlet" + "spin-0 octet"

follows easily. Finally, the IR of SU_6 with 70 dimensions is the IR $\{2, 1\}$. Using Eq. (3.7) and then Eq. (4.32), we get

$$\chi(\{2,1\}) = \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix}$$

$$= \chi_s(\{2\})\chi_f(\{2\})\chi_s(\{1\})\chi_f(\{1\}) + \chi_s(\{0\})\chi_f(\{1,1\})\chi_s(\{1\})\chi_f(\{1\}) - \chi_s(\{3\})\chi_f(\{3\}) - \chi_s(\{1\})\chi_f(\{2,1\}), \tag{4.37}$$

and hence,

$$\{2, 1\} \to \{1\} \otimes \{3\} + \{3\} \otimes \{2, 1\} + \{1\} \otimes \{2, 1\} + \{1\} \otimes \{0\}; \tag{4.38}$$

"70" → "spin-$\frac{1}{2}$ decuplet" + "spin-$\frac{3}{2}$ octet" + "spin-$\frac{1}{2}$ octet" + "spin-$\frac{1}{2}$ singlet."

Results (4.33), (4.36), and (4.38) agree with reductions quoted[5] in the literature.

Table I shows reductions for the lowest IRs of SU_6. The dimensionality of $\{l_1, l_2, l_3, l_4, l_5\}$ is obtained from the formula[12]

$$d(\{l_1, l_2, l_3, l_4, l_5\}) = [(l_1 - l_2 + 1)(l_2 - l_3 + 1) \times (l_3 - l_4 + 1)(l_4 - l_5 + 1)(l_5 + 1) \times (l_1 - l_3 + 2)(l_2 - l_4 + 2)(l_3 - l_5 + 2)(l_4 + 2) \times (l_1 - l_4 + 3)(l_2 - l_5 + 3)(l_3 + 3)(l_1 - l_5 + 4) \times (l_2 + 4)(l_1 + 5)]/(1!2!3!4!5!). \tag{4.39}$$

It is to be noted that the IRs of $SU_2 \otimes SU_3$ contained in an IR of SU_6 of plurality p_6 have duality and triality equal to p_6 mod 2 and p_6 mod 3, respectively, in accord with the theorem of Sec. II.

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[24] Such series may be derived by the Littlewood method[15] [originally applied to SU_3 by A. R. Edmonds, Proc. Roy. Soc. (London) **A268**, 567 (1962)], or by Speiser's method: [D. Speiser, *Proceedings of the Istanbul Summer School, 1962*, (Gordon and Breach, Science Publishers, New York, to be published); J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963)] or by use of tensor methods [N. Mukunda and L. K. Pandit, preprint (1964); S. Coleman, J. Math. Phys. 5, 1343 (1964)]. A general Clebsch–Gordan series has been obtained by various methods [M. Moshinsky, Ref. 23; A. Simoni and B. Vitale, Nuovo Cimento **33**, 1199 (1964); B. Preziosi, A. Simoni, and B. Vitale, Nuovo Cimento **34**, 1101 (1964); L. K. Pandit and N. Mukunda, preprint (1964)].

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Reduction of Representations of SU_{m+n} with respect to the Subgroup $SU_m \otimes SU_n$

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A direct and rapid method is described for the reduction of representations of the special unitary group SU_{m+n} with respect to its $SU_m \otimes SU_n$ subgroup. The technique is illustrated by specific reference to the case of SU_6 and its $SU_2 \otimes SU_4$ subgroup in the context of the recent extension of Wigner's (SU_4) supermultiplet theory into the domain of particle physics. A tabulation of results is given for a slight generalization of this case where reduction is carried out with respect to $U_1 \otimes SU_2 \otimes SU_4$.

I. INTRODUCTION

In the preceding paper by the present authors,[1] a study was made of the branching rules for the special unitary group SU_{mn} with respect to its $SU_m \otimes SU_n$ subgroup; i.e., it was shown how one can determine which irreducible representations of $SU_m \otimes SU_n$ occur in a given irreducible representation of SU_{mn}. This investigation was, of course, undertaken principally as a consequence of the great interest[2] currently associated with the SU_6 theory of Gürsey, Radicati,[3] Pais,[4] and Sakita,[5] in which the spin-unitary spin $SU_2 \otimes SU_3$ subgroup of SU_6 occupies a vital position, and an extensive tabulation of results for this case was given in HM. However, later developments of the SU_6 theory[6] also employ a different subgroup (the so-called unphysical subgroup $U_1 \otimes SU_2 \otimes SU_4$) whose precise significance in relation to the physical $SU_2 \otimes SU_3$ subgroup is reviewed below. It is thus becoming increasingly clear that branching rules for the irreducible representations of SU_6 with respect to this unphysical subgroup may also be expected to be of considerable interest.

We consider here the derivation of branching rules for the special unitary group SU_{m+n} with respect to its $SU_m \otimes SU_n$ subgroup. For the special case $m = 2$, $n = 4$ the method yields a prescription for the determination of the representations of $SU_2 \otimes SU_4$ contained in a given representation of SU_6. It is relatively straightforward to extend these results to the complete subgroup $U_1 \otimes SU_2 \otimes SU_4$. Since U_1 is the hypercharge gauge group (see next paragraph), such extension demands only the specification of the appropriate hypercharge eigenvalue associated with each $SU_2 \otimes SU_4$ representation in a given reduction. Another case for which the general method of the present paper may be relevant is a recent work by Gell-Mann[7] which involves an SU_6 group with an $SU_3 \otimes SU_3$ subgroup. Reductions of the type considered here are also relevant in connection with the work of Moshinsky and his collaborators on Clebsch–Gordon coefficients of unitary groups.[7a]

We may best exhibit the relation (as well as the contrast) between the present and the preceding paper[1] by referring to the SU_6 theory[3-6] together with its physical and unphysical subgroups. Consider then the basis states of the defining representation of SU_6 to be the six states ϕ_{ij} ($1 \leq i \leq 2$, $1 \leq j \leq 3$) of a spin-$\frac{1}{2}$-quark with baryon number $\frac{1}{3}$. Here the label i refers to ordinary spin, ($i = 1$ to spin up and $i = 2$ to spin down) and the label j refers to "unitary spin" ($j = 1$ to the $I = I_z = \frac{1}{2}$, $Y = \frac{1}{3}$ state, $j = 2$ to the $I = -I_z = \frac{1}{2}$, $Y = \frac{1}{3}$ state, and $j = 3$ to the $I = 0$, $Y = -\frac{2}{3}$ state). The physical $SU_2 \otimes SU_3$ subgroup is realized by considering those transformations U of SU_6 which are of the form $U' \otimes U''$, where U' belongs to SU_2 and acts only on the spin label i, and U'' belongs to SU_3 and acts on the unitary spin label j of ϕ_{ij}. To obtain the unphysical subgroup, consider those elements of SU_6 which do not mix states of different

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† Research supported by the U. S. Atomic Energy Commission under Contract AT(30-1)-3399.
[1] C. R. Hagen and A. J. Macfarlane, J. Math. Phys. **6**, 1355 (1965), previous paper. This paper is referred to as HM in the present work.
[2] An extensive list of references to the SU_6 theory may be found in HM.
[3] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964).
[4] A. Pais, Phys. Rev. Letters **13**, 175 (1964).
[5] B. Sakita, Phys. Rev. Letters **13**, 418 (1964); and Phys. Rev. **136**, B1756 (1964).
[6] F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).
[7] M. Gell-Mann, Physics **1**, 63 (1964).
[7a] M. Moskinsky, J. Math. Phys. **4**, 1128 (1963); and T. A. Brody, M. Moshinsky, and I. Renero, *ibid.* (to be published).

Y, i.e., which transform the states ϕ_{i3} ($i = 1, 2$) and the states ϕ_{ij} ($1 \leq i, j \leq 2$) independently among themselves. Writing the ϕ_{ij} now in the order $\phi_{13}, \phi_{23}, \phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$, the elements of the $SU_2 \otimes SU_4$ subgroup of SU_6 are of the form $U' \oplus U''$, where U' belongs to the SU_2 group of spin transformations of the $Y = -\frac{2}{3}$ or $S = -1$ (strangeness-bearing) quark states, and U'' belongs to the SU_4 group of transformations of the spin, isospin states of the $Y = \frac{1}{3}$ or $S = 0$ (non-strangeness-bearing) quark states. It is to be noted that the SU_4 group in question is of the same type as the SU_4 group of the Wigner supermultiplet theory[8] and indeed the interest of the unphysical subgroup of SU_6 is closely related to this fact. Finally, it can be seen that hypercharge gauge transformations of the six basic states ϕ_{ij} commute with the $SU_2 \otimes SU_4$ group so that we may realize $U_1 \otimes SU_2 \otimes SU_4$, where U_1 is hypercharge gauge group, as the complete unphysical subgroup of SU_6.

The approach used here for the derivation of branching rules for SU_{m+n} with respect to its $SU_m \otimes SU_n$ subgroup closely parallels the approach of our previous paper,[1] despite the very different ways in which the product subgroups are imbedded in the parent group. The same concise versions of the Weyl character formula for special unitary groups are employed in an essentially identical fashion to achieve a direct rapid method of calculation rather than a general formula. Details are presented in the next section, with particular emphasis being given to the SU_6 and $SU_2 \otimes SU_4$ groups. In the case of $U_1 \otimes SU_2 \otimes SU_4$ an extensive tabulation of results is appended.

II. DERIVATION OF BRANCHING RULES

We begin with a brief review of some essential notation introduced in Sec. III of HM together with certain results regarding characters of the unitary group SU_p.

The irreducible representations (IRs) of SU_p can be put into one-to-one correspondence with Young diagrams of no more than $(p-1)$ rows and, hence, are characterized by the notation[9]

$$\{l_1, l_2, \cdots, l_{p-1}\}, \tag{2.1}$$

where l_i gives the number of boxes of the ith row of the Young diagram and satisfies

$$l_1 \geq l_2 \geq \cdots \geq l_{p-1} \geq 0. \tag{2.2}$$

[8] E. P. Wigner, Phys. Rev. **51**, 106 (1937); E. P. Wigner and E. Feenberg, Rept. Progr. Phys. **8**, 274 (1941).

[9] Connection with the highest weight notation is explained in C. R. Hagen and A. J. Macfarlane, J. Math. Phys. **5**, 1335 (1964), footnote 11.

We denote by $\chi(\{l_1, l_2, \cdots l_{p-1}\}, \epsilon_1, \epsilon_2, \cdots \epsilon_p)$ the character of the IR (2.1) of SU_p corresponding to an element of SU_p with eigenvalues ϵ_i ($1 \leq i \leq p$) where $|\epsilon_i| = 1$ and $\epsilon_1\epsilon_2 \cdots \epsilon_p = 1$. We wish to use certain concise formulas given in HM which express χ in terms of fundamental symmetric functions of the ϵ_i. These are the so called elementary symmetric functions

$$\begin{aligned} a_1 &= \sum_i \epsilon_i, \\ a_2 &= \sum_{i<j} \epsilon_i\epsilon_j, \\ a_3 &= \sum_{i<j<k} \epsilon_i\epsilon_j\epsilon_k, \\ &\vdots \\ a_p &= \epsilon_1\epsilon_2 \cdots \epsilon_p = 1, \end{aligned} \tag{2.3}$$

and the homogeneous product sums

$$\begin{aligned} h_1 &= \sum_i \epsilon_i = a_1, \\ h_2 &= \sum_{i\leq j} \epsilon_i\epsilon_j, \\ h_3 &= \sum_{i\leq j\leq k} \epsilon_i\epsilon_j\epsilon_k, \\ &\vdots \end{aligned} \tag{2.4}$$

We further define $a_0 = 1$, $a_r = 0$ for $r < 0$ and $r > p$, $h_0 = 1$ and $h_r = 0$ for $r < 0$. It is to be noted that while h_r is defined by (2.4) for all $r > 0$, the h_r for $r > p$ are algebraically dependent on the h_r for $r \leq p$. The relationship of the a_i and the h_i and some of their properties have been discussed in Sec. III of HM, the well-known book by Littlewood[10] having been the source of most of the material in question. In terms of the h_i, we have the result

$$\chi(\{l_1, l_2, \cdots l_{p-1}\}, \epsilon_1, \cdots \epsilon_p) = \begin{vmatrix} h_{l_1} & h_{l_1+1} & \cdots & h_{l_1+p-2} \\ h_{l_2-1} & h_{l_2} & \cdots & h_{l_2+p-3} \\ \vdots & & & \\ h_{l_{p-1}-p+2} & & \cdots & h_{l_{p-1}} \end{vmatrix} \tag{2.5}$$

and in terms of the a_i, we have

$$\chi(\{l_1, l_2, \cdots, l_{p-1}\}, \epsilon_1, \cdots \epsilon_p) = \begin{vmatrix} a_{\lambda_1} & a_{\lambda_1+1} & \cdots & a_{\lambda_1+q-1} \\ a_{\lambda_2-1} & a_{\lambda_2} & \cdots & a_{\lambda_2+q-2} \\ \vdots & & & \\ a_{\lambda_q-q+1} & & \cdots & a_{\lambda_q} \end{vmatrix}, \tag{2.6}$$

[10] D. E. Littlewood, *Theory of Group Characters* (Oxford University Press, London, 1950), 2nd ed., especially Chaps. 5 and 6.

where $\{\lambda_1, \lambda_2, \cdots, \lambda_q\}$ is the Young diagram conjugate to the Young diagram $\{l_1, l_2, \cdots, l_{p-1}\}$; i.e., λ_j is the number of boxes in the jth column of the latter Young diagram. We note (cf. HM, Sec. III) that q is not necessarily less than $(p-1)$ and that (2.6) should be used whenever the IR [Eq. (2.1)] of SU_p has fewer columns than rows. We further remark (cf. HM, Sec. III) that, for IRs $\{l, 0, \cdots 0\}$ (totally symmetric lth rank tensors), Eq. (2.5) reduces to

$$\chi(\{l\}, \epsilon_1, \epsilon_2, \cdots \epsilon_p) = h_l \tag{2.7}$$

and that for the jth fundamental IR $\{1^j, 0, \cdots 0\} \equiv \{1^j\}$, $1 \leq j \leq p-1$ [Eq. (2.6)] reduces to

$$\chi(\{1^j\}, \epsilon_1, \epsilon_2, \cdots \epsilon_p) = a_j. \tag{2.8}$$

We now turn to the question of branching rules for SU_{m+n} with respect to its subgroup $SU_m \otimes SU_n$. To this end, we consider elements U of SU_{m+n} of the form

$$U = \begin{bmatrix} U' & 0 \\ 0 & U'' \end{bmatrix}, \tag{2.9}$$

where U', U'' are, respectively, elements of SU_m and SU_n. Thus to reduce the IR $\{l_1, l_2, \cdots, l_{m+n-1}\}$ with respect to $SU_m \otimes SU_n$, we must impose the restrictions

$$\epsilon_1\epsilon_2 \cdots \epsilon_m = 1, \tag{2.10a}$$

$$\epsilon_{m+1}\epsilon_{m+2} \cdots \epsilon_{m+n} = 1 \tag{2.10b}$$

on the ϵ_1 which appear in its character formula. Then, by expressing the homogenous product sums h_i of the ϵ_a $(1 \leq a \leq m+n)$, subject to (2.10), in terms of the homogeneous product sums h_j' of the ϵ_b $(1 \leq b \leq m)$, subject to (2.10a), and the homogeneous product sums h_k'' of the ϵ_c $(m+1 \leq c \leq m+n)$, subject to (2.10b), one can use the Clebsch–Gordan[11] series for SU_m and SU_n in the expansion of the determinant (2.5) with $p = m+n$ to obtain the desired reduction of the IR of SU_{m+n} in question. Of course, if the IR of SU_{m+n} has fewer columns than rows, one would express a_i in terms of a_j' and a_k'' (in obvious notation) and use Eq. (2.6) rather than (2.5). To obtain the expressions for h_i and a_i alluded to in the previous sentences, we may refer directly to Littlewood for the results[12]

$$h_r = \sum_{i=0}^{r} h'_{r-i}h''_i, \qquad 0 \leq r \leq \infty, \tag{2.11}$$

$$a_r = \sum_{i=0}^{r} a'_{r-i}a''_i, \qquad 0 \leq r \leq m+n. \tag{2.12}$$

We recall, in connection with the latter equation, that $a_i' = 0$ for $i > m$, and $a_i'' = 0$ for $i > n$. The results given are sufficient for a direct treatment to be made of any IR of SU_{m+n}. We further note the following general result given by Littlewood[13]: The number of times the IR $\Gamma' \otimes \Gamma''$ of $SU_m \otimes SU_n$ is contained in the IR Γ of SU_{m+n} equals the number of times Γ occurs in the direct product of those IRS of SU_{m+n} which correspond, respectively, to the same Young diagrams as Γ' and Γ''. While this result allows an easy check on any statement made regarding the $SU_m \otimes SU_n$ content of an IR of SU_{m+n}, it does not seem possible to use it as the basis of a rapid direct approach to derivation of branching rules.[14]

It will be convenient to illustrate our method with reference to the $SU_2 \otimes SU_4$ subgroup of SU_6, returning later to those slight extensions of the method needed for the $U_1 \otimes SU_2 \otimes SU_4$ subgroup. We first consider the fundamental IRs of SU_6. From (2.12) we have, in the same notation as employed above in the general discussion,

$$\begin{aligned} a_1 &= a_1' + a_1'', \\ a_2 &= 1 + a_1'a_1'' + a_2'', \\ a_3 &= a_1'' + a_1'a_2'' + a_3'' = a_3^*, \\ a_4 &= a_2'' + a_1'a_3'' + 1 = a_2^*, \\ a_5 &= a_3'' + a_1' = a_1^*, \end{aligned} \tag{2.13}$$

since $a_0' = a_2' = a_0'' = a_4'' = 1$, $a_1' = a_1'^*$, $a_1'' = a_3''^*$ and $a_2'' = a_2''^*$. Hence, from Eq. (2.8) we deduce the branching rules

$$\begin{aligned} &\{1\} \to \{1\} \otimes \{0\} + \{0\} \otimes \{1\}, \\ &\{1^2\} \to \{0\} \otimes \{0\} + \{1\} \otimes \{1\} + \{0\} \otimes \{1^2\}, \\ &\{1^3\} \to \{0\} \otimes \{1\} + \{1\} \otimes \{1^2\} + \{0\} \otimes \{1^3\}, \\ &\{1^4\} \to \{0\} \otimes \{1^2\} + \{1\} \otimes \{1^3\} + \{0\} \otimes \{0\}, \\ &\{1^5\} \to \{0\} \otimes \{1^3\} + \{1\} \otimes \{0\} \end{aligned} \tag{2.14}$$

for the fundamental IRs of SU_6. On the right of Eq. (2.14) (and hereafter) the first part refers to the SU_2 IR, with $l = 2j$ giving the connection of $\{l\}$ to the corresponding angular momentum quantum number, while the second part refers to the SU_4 part.

To illustrate the general method, we treat the 56- , 35- , and 70-dimensional IRs of SU_6 which have featured centrally in physical discussions of

[11] Such series can be obtained for any unitary group by the Young diagram method; see D. E. Littlewood, Ref. 10, p. 94.

[12] D. E. Littlewood, Ref. 10, p. 104.

[13] D. E. Littlewood, Ref. 10, p. 105.

[14] M. Whippman in a recent investigation of branching rules (preprint, 1964) has also given the result and indicated that it can be made the basis of a recursive derivation of results such as those derived in the present work. See also A. J. Coleman, unpublished notes of lectures given at University of Uppsala, Uppsala, Sweden, 1963.

the SU_6 theory. The 56-dimensional IR is $\{3\}$ and from (2.7) and (2.11) we deduce immediately

$$\{3\} \to \{3\} \otimes \{0\} + \{2\} \otimes \{1\} + \{1\} \otimes \{2\} + \{0\} \otimes \{3\}, \tag{2.15}$$

$$56 = (4 \times 1) + (3 \times 4) + (2 \times 10) + (1 \times 20),$$

the lower line being a check by dimensionalities. The 35-dimensional IR is the regular representation $\{2, 1, 1, 1, 1\}$. Using Eq. (2.6) and Eq. (2.13), we get

$$\chi(\{2, 1, 1, 1, 1\}) = \begin{vmatrix} a_5 & a_6 \\ 1 & a_1 \end{vmatrix} = (a_1' + a_1'')(a_1' + a_3'') - 1, \tag{2.16}$$

and hence, using simple CG series[10] for SU_2 and SU_4, we find directly

$$\{2, 1, 1, 1, 1\} \to \{2\} \otimes \{0\} + \{1\} \otimes \{1^3\} + \{1\} \otimes \{1\} + \{0\} \otimes \{0\} + \{0\} \otimes \{2, 1, 1\} \tag{2.17}$$

with the check

$$35 = (3 \times 1) + (2 \times 4) + (2 \times 4) + (1 \times 1) + (1 \times 15)$$

by dimensionalities. The 70-dimensional IR is $\{2, 1\}$, and we use Eqs. (2.5) and (2.11) to find

$$\chi(\{2, 1\}) = \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix} = (h_2' + h_1'h_1'' + h_2'')(h_1' + h_1'') - (h_3' + h_2'h_1'' + h_1'h_2'' + h_3''). \tag{2.18}$$

Hence, with the aid of Eq. (2.7) and simple CG series,[11] we obtain the result

$$\{2, 1\} \to \{1\} \otimes \{0\} + \{2\} \otimes \{1\} + \{0\} \otimes \{1\} + \{1\} \otimes \{2\} + \{1\} \otimes \{1^2\} + \{0\} \otimes \{2, 1\} \tag{2.19}$$

with the check

$$70 = (2 \times 1) + (3 \times 4) + (1 \times 4) + (2 \times 10) + (2 \times 6) + (1 \times 20)$$

by dimensionalities.

In order to obtain reductions for IRs of SU_6 with respect to the full unphysical subgroup, we must make a slight extension of the previous considerations, since the unphysical subgroup is not $SU_2 \otimes SU_4$ but rather $U_1 \otimes SU_2 \otimes SU_4$. The U_1 gauge group comes in, as explained in the introduction, because of the existence of the hypercharge gauge group which commutes with the $SU_2 \otimes SU_4$ subgroup of SU_6. In terms of the ordering ϕ_{13}, ϕ_{23}, ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} of basis states of the defining representation, we can write hypercharge transformations in the form

$$e^{iY\theta} = \eta^{-\frac{2}{3}}\mathbb{1} \oplus \eta^{\frac{1}{3}}\mathbb{1}, \tag{2.20}$$

where

$$\eta = e^{i\theta}, \qquad Y = -\tfrac{2}{3}\mathbb{1} \oplus \tfrac{1}{3}\mathbb{1}, \tag{2.21}$$

which clearly exhibits commutativity with the elements $U = U' \oplus U''$ of $SU_2 \otimes SU_4$. The reduction of IRs of SU_6 with respect to $U_1 \otimes SU_2 \otimes SU_4$ is simply effected by the association of an appropriate hypercharge eigenvalue with each IR of $SU_2 \otimes SU_4$ encountered in the application of the methods of the previous paragraph. We can, however, easily determine the appropriate hypercharge eigenvalues by a simple extension of the previous method. To this end we consider elements of SU_6 of the form $\mathcal{U} = e^{iY\theta}U$, where $e^{iY\theta}$ is given by Eq. (2.20) and $U = U' \oplus U''$ is an element of $SU_2 \otimes SU_4$ of the type discussed above. The eigenvalues $\bar{\epsilon}_i$ $(1 \le i \le 6)$ of $\mathcal{U}$ can be written as

$$\bar{\epsilon}_i = \epsilon_i \eta^{-\frac{2}{3}} \quad (i = 1, 2) \qquad \text{with} \qquad \epsilon_1\epsilon_2 = 1, \tag{2.22}$$

$$\bar{\epsilon}_i = \epsilon_i \eta^{\frac{1}{3}} \quad (i = 3, 4, 5, 6) \quad \text{with} \quad \epsilon_3\epsilon_4\epsilon_5\epsilon_6 = 1, \tag{2.23}$$

and we must express the homogeneous product sums h_r and the elementary symmetric functions a_r of the $\bar{\epsilon}_i$ in terms of η and the same quantities h', h'', a', and a'', as occur above. The desired expressions easily follow from the $m = 2$, $n = 4$ cases of Eqs. (2.11) and (2.12), since h, h', h'', a, a', and a'' are homogeneous functions of their arguments. They are readily found to be

$$h_r = \sum_{i=0}^{r} h_{r-i}'h_i''\eta^{-\frac{2}{3}r+i} \quad (r \ge 0), \tag{2.24}$$

$$a_r = \sum_{i=0}^{r} a_{r-i}'a_i''\eta^{-\frac{2}{3}r+i}. \tag{2.25}$$

Clearly, we can use Eq. (2.24) along with Eq. (2.5) in the present case exactly as we used (the $m = 2$, $n = 4$ version of) Eq. (2.11) along with Eq. (2.5) in the previous paragraph. The only difference is that associated with each term of the final simplified character formula we have now a power of η whose exponent gives the hypercharge eigenvalue of the corresponding $SU_2 \otimes SU_4$ representation. It is clear that an entirely analogous statement applies to the formulas based on the functions a_r.

A table of results for various IRs of SU_6 (low-dimensional ones or those used in the discussion of the physical group SU_6/Z_3) is appended. The dimensionality formula for SU_6 is

$$D(\{l_1, l_2, l_3, l_4, l_5\}) = [(l_1 - l_2 + 1)(l_2 - l_3 + 1) \times (l_3 - l_4 + 1)(l_4 - l_5 + 1)(l_5 + 1)$$

TABLE I. Hexalities of IRs of SU_6 and the reduction with respect to $U_1 \otimes SU_2 \otimes SU_4$.

IR of SU_6	Dimensionality	Hexality	$U_1 \otimes SU_2 \otimes SU_4$ content
$\{1\}$	6	1	$(-\frac{2}{3})\{1\} \otimes \{0\} + (\frac{1}{3})\{0\} \otimes \{1\}$
$\{1, 1\}$	15	2	$(-\frac{4}{3})\{0\} \otimes \{0\} + (-\frac{1}{3})\{1\} \otimes \{1\} + (\frac{2}{3})\{0\} \otimes \{1, 1\}$
$\{1, 1, 1\}$	20	3	$(-1)\{0\} \otimes \{1\} + (0)\{1\} \otimes \{1, 1\} + (1)\{0\} \otimes \{1, 1, 1\}$
$\{1, 1, 1, 1\}$	15	4	$(-\frac{2}{3})\{0\} \otimes \{1, 1\} + (\frac{1}{3})\{1\} \otimes \{1, 1, 1\} + (\frac{4}{3})\{0\} \otimes \{0\}$
$\{1, 1, 1, 1, 1\}$	6	5	$(-\frac{1}{3})\{0\} \otimes \{1, 1, 1\} + (\frac{2}{3})\{1\} \otimes \{0\}$
$\{2\}$	21	2	$(-\frac{4}{3})\{2\} \otimes \{0\} + (\frac{2}{3})\{0\} \otimes \{2\} + (-\frac{1}{3})\{1\} \otimes \{1\}$
$\{2, 1, 1, 1, 1\}$	35	0	$(0)\{2\} \otimes \{0\} + (-1)\{1\} \otimes \{1, 1, 1\} + (1)\{1\} \otimes \{1\} + (0)\{0\} \otimes \{0\} + (0)\{0\} \otimes \{2, 1, 1\}$
$\{3\}$	56	3	$(-2)\{3\} \otimes \{0\} + (-1)\{2\} \otimes \{1\} + (0)\{1\} \otimes \{2\} + (1)\{0\} \otimes \{3\}$
$\{2, 1\}$	70	3	$(-2)\{1\} \otimes \{0\} + (-1)\{2\} \otimes \{1\} + (-1)\{0\} \otimes \{1\} + (0)\{1\} \otimes \{2\} + (0)\{1\} \otimes \{1, 1\} + (1)\{0\} \otimes \{2, 1\}$
$\{2, 1, 1, 1\}$	84	5	$(-\frac{4}{3})\{1\} \otimes \{1, 1\} + (-\frac{1}{3})\{2\} \otimes \{1, 1, 1\} + (\frac{5}{3})\{0\} \otimes \{1\} + (-\frac{1}{3})\{0\} \otimes \{1, 1, 1\} + (\frac{2}{3})\{1\} \otimes \{0\} + (\frac{2}{3})\{1\} \otimes \{2, 1, 1\} + (-\frac{1}{3})\{0\} \otimes \{2, 1\}$
$\{2, 2\}$	105	4	$(-\frac{8}{3})\{0\} \otimes \{0\} + (-\frac{5}{3})\{1\} \otimes \{1\} + (-\frac{2}{3})\{0\} \otimes \{1, 1\} + (-\frac{2}{3})\{2\} \otimes \{2\} + (\frac{4}{3})\{0\} \otimes \{2, 2\} + (\frac{1}{3})\{1\} \otimes \{2, 1\}$
$\{2, 1, 1\}$	105	4	$(-\frac{5}{3})\{1\} \otimes \{1\} + (-\frac{2}{3})\{2\} \otimes \{1, 1\} + (\frac{4}{3})\{0\} \otimes \{2, 1, 1\} + (-\frac{2}{3})\{0\} \otimes \{2\} + (-\frac{2}{3})\{0\} \otimes \{1, 1\} + (\frac{1}{3})\{1\} \otimes \{2, 1\} + (\frac{1}{3})\{1\} \otimes \{1, 1, 1\}$
$\{4\}$	126	4	$(-\frac{8}{3})\{4\} \otimes \{0\} + (-\frac{5}{3})\{3\} \otimes \{1\} + (-\frac{2}{3})\{2\} \otimes \{2\} + (\frac{1}{3})\{1\} \otimes \{3\} + (\frac{4}{3})\{0\} \otimes \{4\}$
$\{2, 2, 2\}$	175	0	$(-2)\{0\} \otimes \{2\} + (-1)\{1\} \otimes \{2, 1\} + (0)\{0\} \otimes \{2, 1, 1\} + (0)\{2\} \otimes \{2, 2\} + (2)\{0\} \otimes \{2, 2, 2\} + (1)\{1\} \otimes \{2, 2, 1\}$
$\{2, 2, 1, 1\}$	189	0	$(-2)\{0\} \otimes \{1, 1\} + (-1)\{1\} \otimes \{1, 1, 1\} +(0)\{0\} \otimes \{0\} + (-1)\{1\} \otimes \{2, 1\} + (0)\{2\} \otimes \{2, 1, 1\} + (0)\{0\} \otimes \{2, 1, 1\} + (1)\{1\} \otimes \{1\} + (0)\{0\} \otimes \{2, 2\} + (2)\{0\} \otimes \{1, 1\} + (1)\{1\} \otimes \{2, 2, 1\}$
$\{3, 1\}$	210	4	$(-\frac{8}{3})\{2\} \otimes \{0\} + (-\frac{5}{3})\{3\} \otimes \{1\} + (-\frac{5}{3})\{1\} \otimes \{1\} + (-\frac{2}{3})\{2\} \otimes \{1, 1\} + (-\frac{2}{3})\{2\} \otimes \{2\} + (-\frac{2}{3})\{0\} \otimes \{2\} + (\frac{1}{3})\{1\} \otimes \{3\} + (\frac{1}{3})\{1\} \otimes \{2, 1\} + (\frac{4}{3})\{0\} \otimes \{3, 1\}$
$\{2, 2, 1\}$	210	5	$(-\frac{7}{3})\{0\} \otimes \{1\} + (-\frac{1}{3})\{0\} \otimes \{1, 1, 1\} + (-\frac{4}{3})\{1\} \otimes \{2\} + (-\frac{4}{3})\{1\} \otimes \{1, 1\} + (-\frac{1}{3})\{2\} \otimes \{2, 1\} + (\frac{2}{3})\{1\} \otimes \{2, 1, 1\} + (-\frac{1}{3})\{0\} \otimes \{2, 1\} + (\frac{2}{3})\{1\} \otimes \{2, 2\} + (\frac{5}{3})\{0\} \otimes \{2, 2, 1\}$
$\{5\}$	252	5	$(-\frac{10}{3})\{5\} \otimes \{0\} + (-\frac{1}{3})\{2\} \otimes \{3\} + (-\frac{4}{3})\{3\} \otimes \{2\} + (\frac{5}{3})\{0\} \otimes \{5\} + (-\frac{7}{3})\{4\} \otimes \{1\} + (\frac{2}{3})\{1\} \otimes \{4\}$

$$\times (l_1 - l_3 + 2)(l_2 - l_4 + 2)(l_3 - l_5 + 2)$$
$$\times (l_4 + 2)(l_1 - l_4 + 3)(l_2 - l_5 + 3)(l_3 + 3)$$
$$\times (l_1 - l_5 + 4)(l_2 + 4)(l_1 + 5)](1!2!3!4!5!)^{-1}, \quad (2.26)$$

while those for SU_2 and SU_4 are

$$D(\{l\}) = l + 1,$$

$$D(\{l_1, l_2, l_3\})$$
$$= [(l_1 - l_2 + 1)(l_2 - l_3 + 1)(l_3 + 1) \quad (2.27)$$
$$\times (l_1 - l_3 + 2)(l_2 + 2)(l_1 + 3)](1!2!3!)^{-1}, \quad (2.28)$$

respectively. These formulas can be used as above to give a check by dimensionalities.[15] A further check on these results is provided by the pluralities[9] of the IRs of SU_6, SU_4, and SU_2 as expressed by the "equation"

$$\text{hexality of } SU_6 \text{ (mod 2)} - \text{duality of } SU_2 - \text{quadrality of } SU_4 \text{ (mod 2)} = 0 \text{ (mod 2)}. \quad (2.29)$$

Finally, we note that on transcription to the highest weight notation $(\alpha, \beta, \gamma, \delta, \epsilon)$[9] for IRs of SU_6, the results of Table I can be readily used to determine the reduction of the complex-conjugate IRs $(\epsilon, \delta, \gamma, \beta, \alpha)$.

[15] A further partial check of results can be attained as follows. From the $SU_2 \otimes SU_3$ reduction of SU_6 IRs (see Table I of HM) and the known[16] Y content of SU_3 IRs, we may obtain the multiplicity of each Y value in any given IR of SU_6. This can be compared with what follows from Table I of the present work and use of known dimensionalities for SU_2 and SU_4 IRs.

[16] C. R. Hagen and A. J. Macfarlane, J. Math. Phys. 5, 1335 (1964).

TABLE I. (*Continued*).

IR of SU_6	Dimensionality	Hexality	$U_1 \otimes SU_2 \otimes SU_4$ content
{3, 1, 1, 1}	280	0	$(-2)\{2\}\otimes\{1,1\}+(-1)\{3\}\otimes\{1,1,1\}+(-1)\{1\}\otimes\{1,1,1\}+(-1)\{1\}\otimes\{2,1\}+(0)\{2\}\otimes\{0\}+(0)\{2\}\otimes\{2,1,1\}+(2)\{0\}\otimes\{2\}+(0)\{0\}\otimes\{2,1,1\}+(0)\{0\}\otimes\{3,1\}+(1)\{1\}\otimes\{3,1,1\}+(1)\{1\}\otimes\{1\}$
{4, 2, 2, 2, 2}	405	0	$(0)\{4\}\otimes\{0\}+(-1)\{3\}\otimes\{1,1,1\}+(1)\{3\}\otimes\{1\}+(-1)\{1\}\otimes\{1,1,1\}+(0)\{0\}\otimes\{2,1,1\}+(2)\{2\}\otimes\{2\}+(0)\{2\}\otimes\{0\}+(0)\{2\}\otimes\{2,1,1\}+(1)\{1\}\otimes\{1\}+(1)\{1\}\otimes\{3,1,1\}+(-2)\{2\}\otimes\{2,2,2\}+(-1)\{1\}\otimes\{3,2,2\}+(0)\{0\}\otimes\{4,2,2\}+(0)\{0\}\otimes\{0\}$
{6}	462	0	$(-4)\{6\}\otimes\{0\}+(-1)\{3\}\otimes\{3\}+(-2)\{4\}\otimes\{2\}+(-3)\{5\}\otimes\{1\}+(0)\{2\}\otimes\{4\}+(2)\{0\}\otimes\{6\}+(1)\{1\}\otimes\{5\}$
{3, 3}	490	0	$(-4)\{0\}\otimes\{0\}+(-3)\{1\}\otimes\{1\}+(-2)\{2\}\otimes\{2\}+(-2)\{0\}\otimes\{1,1\}+(-1)\{3\}\otimes\{3\}+(0)\{2\}\otimes\{3,1\}+(-1)\{1\}\otimes\{2,1\}+(2)\{0\}\otimes\{3,3\}+(1)\{1\}\otimes\{3,2\}+(0)\{0\}\otimes\{2,2\}$
{3, 2, 2, 1, 1}	540	3	$(1)\{2\}\otimes\{1,1,1\}+(0)\{3\}\otimes\{1,1\}+(-2)\{1\}\otimes\{2,1,1\}+(2)\{1\}\otimes\{2,1,1\}+(1)\{2\}\otimes\{2,1\}+(-1)\{0\}\otimes\{3,1,1\}+(1)\{0\}\otimes\{3,2,2\}+(1)\{0\}\otimes\{2,1\}+(1)\{0\}\otimes\{1,1,1\}+(-1)\{2\}\otimes\{2,2,1\}+(-1)\{0\}\otimes\{2,2,1\}+(-1)\{2\}\otimes\{1\}+(-1)\{0\}\otimes\{1\}+(0)\{1\}\otimes\{2,2,2\}+(0)\{1\}\otimes\{3,2,1\}+(0)\{1\}\otimes\{1,1\}+(0)\{1\}\otimes\{2\}+(0)\{1\}\otimes\{1,1\}$
{3, 2, 2, 2}	560	3	$(1)\{2\}\otimes\{1,1,1\}+(0)\{3\}\otimes\{2,2,2\}+(0)\{1\}\otimes\{2,2,2\}+(-2)\{1\}\otimes\{2,2\}+(-1)\{2\}\otimes\{2,2,1\}+(3)\{0\}\otimes\{1\}+(2)\{1\}\otimes\{2,1,1\}+(2)\{1\}\otimes\{0\}+(1)\{0\}\otimes\{1,1,1\}+(1)\{0\}\otimes\{2,1\}+(1)\{2\}\otimes\{3,2,2\}+(-1)\{0\}\otimes\{3,2\}+(-1)\{0\}\otimes\{2,2,1\}+(0)\{1\}\otimes\{3,2,1\}+(0)\{1\}\otimes\{1,1\}$
{5, 1, 1, 1, 1}	700	3	$(-2)\{5\}\otimes\{0\}+(1)\{2\}\otimes\{3\}+(-3)\{4\}\otimes\{1,1,1\}+(-1)\{4\}\otimes\{1\}+(0)\{3\}\otimes\{2\}+(0)\{1\}\otimes\{2\}+(2)\{1\}\otimes\{4\}+(-2)\{3\}\otimes\{0\}+(1)\{0\}\otimes\{3\}+(-2)\{3\}\otimes\{2,1,1\}+(-1)\{2\}\otimes\{3,1,1\}+(0)\{1\}\otimes\{4,1,1\}+(1)\{0\}\otimes\{5,1,1\}$
{3, 3, 3}	980	3	$(-3)\{0\}\otimes\{3\}+(-2)\{1\}\otimes\{3,1\}+(-1)\{2\}\otimes\{3,2\}+(1)\{0\}\otimes\{3,2,2\}+(0)\{1\}\otimes\{3,2,1\}+(-1)\{0\}\otimes\{3,1,1\}+(0)\{3\}\otimes\{3,3\}+(2)\{1\}\otimes\{3,3,2\}+(1)\{2\}\otimes\{3,3,1\}+(3)\{0\}\otimes\{3,3,3\}$
{5, 1}	1050	0	$(-4)\{4\}\otimes\{0\}+(-1)\{1\}\otimes\{3\}+(-2)\{2\}\otimes\{2\}+(1)\{1\}\otimes\{5\}+(-3)\{3\}\otimes\{1\}+(0)\{2\}\otimes\{4\}+(0)\{0\}\otimes\{4\}+(-3)\{5\}\otimes\{1\}+(0)\{2\}\otimes\{3,1\}+(-1)\{3\}\otimes\{2,1\}+(-1)\{3\}\otimes\{3\}+(2)\{0\}\otimes\{5,1\}+(-2)\{4\}\otimes\{2\}+(-2)\{4\}\otimes\{1,1\}+(1)\{1\}\otimes\{4,1\}$
{4, 2}	1134	0	$(-4)\{2\}\otimes\{0\}+(-3)\{3\}\otimes\{1\}+(-2)\{4\}\otimes\{2\}+(-3)\{1\}\otimes\{1\}+(-1)\{3\}\otimes\{3\}+(-2)\{2\}\otimes\{2\}+(-2)\{2\}\otimes\{1,1\}+(-2)\{0\}\otimes\{2\}+(-1)\{3\}\otimes\{2,1\}+(-1)\{1\}\otimes\{3\}+(-1)\{1\}\otimes\{2,1\}+(0)\{2\}\otimes\{4\}+(0)\{2\}\otimes\{3,1\}+(0)\{2\}\otimes\{2,2\}+(0)\{0\}\otimes\{3,1\}+(1)\{1\}\otimes\{4,1\}+(1)\{1\}\otimes\{3,2\}+(2)\{0\}\otimes\{4,2\}$
{4, 2, 1, 1, 1}	1134	3	$(-2)\{3\}\otimes\{0\}+(-2)\{1\}\otimes\{0\}+(-1)\{4\}\otimes\{1\}+(-1)\{2\}\otimes\{1\}+(-1)\{0\}\otimes\{1\}+(0)\{3\}\otimes\{1,1\}+(0)\{1\}\otimes\{1,1\}+(0)\{3\}\otimes\{2\}+(0)\{1\}\otimes\{2\}\otimes(1)\{2\}\otimes\{3\}+(1)\{0\}\otimes\{3\}+(1)\{2\}\otimes\{2,1\}+(1)\{0\}\otimes\{2,1\}+(2)\{1\}\otimes\{3,1\}+(-3)\{2\}\otimes\{1,1,1\}+(-2)\{3\}\otimes\{2,1,1\}+(-2)\{1\}\otimes\{2,1,1\}+(-1)\{2\}\otimes\{2,2,1\}+(-1)\{2\}\otimes\{3,1,1\}+(-1)\{2\}\otimes\{1\}+(-1)\{0\}\otimes\{3,1,1\}+(0)\{1\}\otimes\{4,1,1\}+(0)\{1\}\otimes\{2\}+(0)\{1\}\otimes\{3,2,1\}+(1)\{0\}\otimes\{4,2,1\}$
{3, 3, 2, 1}	1960	3	$(-3)\{0\}\otimes\{2,1\}+(-2)\{1\}\otimes\{2,2\}+(-2)\{1\}\otimes\{2,1,1\}+(-1)\{0\}\otimes\{3,1,1\}+(-1)\{0\}\otimes\{2,2,1\}+(-1)\{2\}\otimes\{2,2,1\}+(1)\{0\}\otimes\{1,1,1\}+(0)\{1\}\otimes\{2,2,2\}+(-2)\{1\}\otimes\{3,1\}+(-1)\{2\}\otimes\{3,2\}+(-1)\{2\}\otimes\{3,1,1\}+(-1)\{0\}\otimes\{3,2\}+(0)\{1\}\otimes\{3,2,1\}+(0)\{3\}\otimes\{3,2,1\}+(0)\{1\}\otimes\{3,3\}+(0)\{1\}\otimes\{1,1\}+(1)\{2\}\otimes\{3,3,1\}+(1)\{2\}\otimes\{3,2,2\}+(1)\{2\}\otimes\{2,1\}+(1)\{0\}\otimes\{3,3,1\}+(1)\{0\}\otimes\{3,2,2\}+(1)\{0\}\otimes\{2,1\}+(-1)\{0\}\otimes\{1\}+(0)\{1\}\otimes\{3,2,1\}+(0)\{1\}\otimes\{2\}+(3)\{0\}\otimes\{2,2,1\}+(2)\{1\}\otimes\{2,1,1\}+(2)\{1\}\otimes\{3,3,2\}+(2)\{1\}\otimes\{2,2\}$
{6, 3, 3, 3, 3}	2695	0	$(0)\{6\}\otimes\{0\}+(-1)\{5\}\otimes\{1,1,1\}+(-2)\{4\}\otimes\{2,2,2\}+(-3)\{3\}\otimes\{3,3,3\}+(1)\{5\}\otimes\{1\}+(0)\{4\}\otimes\{2,1,1\}+(0)\{4\}\otimes\{0\}+(-1)\{3\}\otimes\{3,2,2\}+(-1)\{3\}\otimes\{1,1,1\}+(-1)\{1\}\otimes\{3,2,2\}+(-1)\{1\}\otimes\{1,1,1\}+(-2)\{2\}\otimes\{4,3,3\}+(-2)\{2\}\otimes\{2,2,2\}+(2)\{4\}\otimes\{2\}+(1)\{3\}\otimes\{3,1,1\}+(1)\{3\}\otimes\{1\}+(1)\{1\}\otimes\{3,1,1\}+(0)\{2\}\otimes\{4,2,2\}+(0)\{2\}\otimes\{2,1,1\}+(0)\{2\}\otimes\{0\}+(0)\{0\}\otimes\{2,1,1\}+(0)\{0\}\otimes\{0\}+(-1)\{1\}\otimes\{5,3,3\}+(3)\{3\}\otimes\{3\}+(2)\{2\}\otimes\{4,1,1\}+(2)\{2\}\otimes\{2\}+(1)\{1\}\otimes\{5,2,2\}+(1)\{1\}\otimes\{1\}+(0)\{0\}\otimes\{6,3,3\}+(0)\{0\}\otimes\{4,2,2\}$

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Symmetrical Coupling of Three Angular Momenta

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We introduce here a new coupling scheme for three angular momenta. It relies on the properties of an operator which depends "democratically" upon the three individual angular momenta; this is in fact their mixed product. This operator, the total angular momentum and one of its components form a complete set of commuting observables. In the case where the three individual angular momenta are equal, the eigenstates of this set possess remarkable symmetry properties with respect to the permutation group S_3.

1. INTRODUCTION

THIS paper deals with an old problem: the quantum mechanical coupling of three angular momenta. Actually, this amounts to reducing into a sum of irreducible representations, a tensor product of three irreducible representations of the three-dimensional rotation group. Such a tensor product is not a multiplicity-free representation; one can find several equivalent irreducible representations in its decomposition. It is thus necessary to introduce a degeneracy parameter which enables one to distinguish these various equivalent irreducible representations. For the physicist, the problem is to find an operator which, added to the three "individual" angular momenta, the total angular momentum and one of its components, gives a complete set of observables.

The well-known solution of this problem rests on the fact that the rotation group is "simply reducible",[1] that is to say, a given irreducible representation shows up once (at most) in the reduction of the tensor product of *two* irreducible representations. When dealing with the tensor product of three such representations, one can then first reduce the tensor product of two among them and for each of the irreducible representations thus obtained, reduce its tensor product with the third of the initial representations. In other words, degeneracy is removed by introducing the angular momentum of a pair of the initial "systems." But this age-old solution is completely "antidemocratic," in that the three initial angular momenta are not treated on the same footing. Now, there are a lot of problems where they play the same part. This is particularly true when one has to couple three equal angular momenta. The representation space of the rotation group reppresentation is then also a representation space for the symmetric group of the third degree. We know that, in a certain sense, we can reduce simultaneously the rotation group and the symmetric group representations.[2] Unhappily, the usual coupling schemes completely obscure this point. In the case of several $\frac{1}{2}$ spins only, do we know explicitly the symmetry properties of the coupled states.[3] And yet, this is a crucial problem when one has to classify the spin or isospin states of several identical particles obeying Fermi–Dirac or Bose–Einstein statistics.

We introduce here a new coupling scheme based upon a new operator with the following properties: (a) its depends "democratically" upon the individual angular momentum operators; (b) it enjoys a simple physical interpretation: indeed, it is the mixed product of the three angular momenta; (c) it is effectively able to remove the degeneracy with respect to the total angular momentum. These properties are derived in Sec. 2.

We then treat in Sec. 3 the case of three equal angular momenta. We show how to reduce completely the relevant representations of the third-degree symmetric group. We carry on this reduction and this provides us with the symmetry properties of the basis vectors in this new coupling scheme.

In the course of our calculations we are led to derive some new properties of Wigner's $6j$ coefficients (see Appendices A and C).

Finally, two tables exhibit the basis vectors of our coupling scheme for the case of three $j = \frac{1}{2}$ and three $j = 1$ angular momenta.

2. THE NEW COUPLING SCHEME

2.1. Definitions

Let $\mathbf{J}^1$, $\mathbf{J}^2$, $\mathbf{J}^3$ be three angular momentum op-

[1] E. P. Wigner, Am. J. Math. **63**, 57 (1941).

[2] See for instance G. Ya. Lubarskii, *The Applications of Group Theory in Physics* (Pergamon Press, Ltd., London, 1960), Chap. IV.

[3] H. V. McIntosh, J. Math. Phys. **1**, 453 (1960) and other references therein.

erators. They are Hermitian operators obeying the following commutation rules:

$$[J_i^\alpha, J_j^\beta] = i\delta^{\alpha\beta}\epsilon_{ijk}J_k^\alpha \quad \begin{cases} \alpha, \beta = 1, 2, 3, \\ i, j, k = 1, 2, 3. \end{cases} \tag{2.1}$$

One defines the total angular momentum operator

$$\mathbf{J} = \mathbf{J}^1 + \mathbf{J}^2 + \mathbf{J}^3. \tag{2.2}$$

We introduce now the Hermitian operator

$$K = \epsilon_{ijk}J_i^1J_j^2J_k^3. \tag{2.3}$$

It can be rewritten as

$$K = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}J_i^\alpha J_j^\beta J_k^\gamma. \tag{2.4}$$

In spherical components, the most useful ones for explicit calculations, this reads

$$K = -\frac{i}{2}\epsilon_{pqr}J_p^1J_q^2J_r^3 \quad \begin{cases} p, q, r = +, 0, -, \\ \epsilon_{+0-} = +1. \end{cases} \tag{2.5}$$

It is straightforward to derive the following commutation rules

$$[K, \mathbf{J}^1] = i[\mathbf{J}^2(\mathbf{J}^1\cdot\mathbf{J}^3) - \mathbf{J}^3(\mathbf{J}^2\cdot\mathbf{J}^1)], \tag{2.6}$$

and cyclically.

As a result,

$$[K, \mathbf{J}] = 0. \tag{2.7}$$

It is then clear that the following set of operators: $(\mathbf{J}^1)^2$, $(\mathbf{J}^2)^2$, $(\mathbf{J}^3)^2$, $(\mathbf{J})^2$, J_0 and K form a commuting set. That this is actually a *complete* set of observables will be shown in the following. Let us note some rather remarkable commutation rules,

$$[\mathbf{J}^1\cdot\mathbf{J}^2, \mathbf{J}^2\cdot\mathbf{J}^3] = \tfrac{1}{4}[(\mathbf{J}^1 + \mathbf{J}^2)^2, (\mathbf{J}^2 + \mathbf{J}^3)^2] = iK, \tag{2.8}$$

and cyclically.

Also,

$$[K, \mathbf{J}^1\cdot\mathbf{J}^2] = i\{(\mathbf{J}^1\cdot\mathbf{J}^2)(\mathbf{J}^3\cdot\mathbf{J}^1) - (\mathbf{J}^2\cdot\mathbf{J}^3)(\mathbf{J}^1\cdot\mathbf{J}^2) + (\mathbf{J}^2)^2(\mathbf{J}^3\cdot\mathbf{J}^1) - (\mathbf{J}^1)^2(\mathbf{J}^2\cdot\mathbf{J}^3)\} \tag{2.9}$$

and cyclically.

2.2. Matrices Elements

Let us now restrict ourselves to the tensor product $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2} \otimes \mathfrak{D}^{j_3}$ of three irreducible representations of the rotation group. We work from now on in the corresponding representation space. In this vector space, the individual operators $(\mathbf{J}^\alpha)^2$ reduce to multiple of the identity:

$$(\mathbf{J}^\alpha)^2 = j_\alpha(j_\alpha + 1)\mathbb{1}, \tag{2.10}$$

so we are no longer interested in them.

Our task is now to solve the simultaneous eigenvalue problem for K, $(\mathbf{J})^2$, and J_0. Maybe, one can solve this democratic problem in a democratic way, for instance by systematically exploiting the commutation rules (2.8) and (2.9). What will be done here, instead, is to use an antidemocratic but straightforward method.

We first notice that we already know the eigenvalues of $(\mathbf{J})^2$ and J_0. The usual coupling schemes enable us to decompose the vector space of the representation in a sum of subspaces $\mathcal{H}_{JM}$ corresponding to the eigenvalues $J(J + 1)$ of $(\mathbf{J})^2$ and M of J_0. These subspaces have, in general, a dimension higher than one due to the fact that the $\mathfrak{D}^J$ representation shows up with a certain multiplicity in the reduction of the tensor product $\mathfrak{D}^{j_1} \otimes \mathfrak{D}^{j_2} \otimes \mathfrak{D}^{j_3}$. But we know a basis in each of these subspaces, obtained by diagonalizing, say, $(\mathbf{J}^1 + \mathbf{J}^2)^2$. This is the usual coupling scheme. Let us note $|j_{12}JM\rangle$ the corresponding normed basis vectors. They are defined by

$$\begin{aligned} (\mathbf{J})^2\,|j_{12}JM\rangle &= J(J+1)\,|j_{12}JM\rangle, \\ J_0\,|j_{12}JM\rangle &= M\,|j_{12}JM\rangle, \\ (\mathbf{J}^1 + \mathbf{J}^2)^2\,|j_{12}JM\rangle &= j_{12}(j_{12}+1)\,|j_{12}JM\rangle, \\ \langle j'_{12}J'M' \mid j_{12}JM\rangle &= \delta_{j_{12}j'_{12}}\delta_{JJ'}\delta_{MM'}. \end{aligned} \tag{2.11}$$

We are going to study the matrix elements of K in this basis. Since K commutes with $(\mathbf{J})^2$ and J_0 we have at once

$$\langle j'_{12}J'M'|\,K\,|j_{12}JM\rangle = \delta_{JJ'}\delta_{MM'}\,\langle j'_{12}JM|\,K\,|j_{12}JM\rangle. \tag{2.12}$$

This proves it suffices to work in each subspace $\mathcal{H}_{JM}$.

Now, K is an irreducible tensor operator (a scalar one, indeed) built up with the three irreducible tensor operators $\mathbf{J}^\alpha$. We have thus at our disposal the standard methods of evaluation of its matrix elements.[4] However, an immediate use of these methods leads to a complicated expression containing one $9j$ and one $6j$ symbols. A little trick simplifies the derivation and leads to an expression containing but $6j$ symbols. The relations (2.8) reads

$$K = \frac{i}{4}[(\mathbf{J}^1 + \mathbf{J}^2)^2, (\mathbf{J}^2 + \mathbf{J}^3)^2], \tag{2.13}$$

whence

$$\langle j'_{12}JM|\,K\,|j_{12}JM\rangle = \frac{i}{4}(j'_{12}(j'_{12}+1) - j_{12}(j_{12}+1))\,\langle j'_{12}JM|\,(\mathbf{J}^2 + \mathbf{J}^3)^2\,|j_{12}JM\rangle. \tag{2.14}$$

[4] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 7.

We evaluate the matrix elements of $(\mathbf{J}^2 + \mathbf{J}^3)^2$ in the Appendix A. We also obtain a new sum formula for $6j$ coefficients. We finally obtain the following most simple result:

$$\langle j'_{12}JM|\, K\, |j_{12}JM\rangle = i\alpha^J_l(\delta_{j'_{12},l}\delta_{j_{12},l-1} - \delta_{j'_{12},l-1}\delta_{j_{12},l}), \tag{2.15}$$

where

$$\alpha^J_l = \tfrac{1}{4}(-1)^{j_1+j_2+j_3+J}[j_2(j_2+1)(2j_2+1)j_3(j_3+1)(2j_3+1)]^{\frac{1}{2}} \times l[(2l-1)(2l+1)]^{\frac{1}{2}}\begin{Bmatrix} J & j_3 & l \\ 1 & l-1 & j_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & l \\ 1 & l-1 & j_2 \end{Bmatrix} \tag{2.16}$$

or

$$\alpha^J_l = \frac{1}{4}\left[\frac{(J+j_3+l+1)(J+j_3-l+1)(j_3-J+l)(J-j_3+l)(j_1+j_2+l+1)(j_1+j_2-l+1)(j_2-j_1+l)(j_1-j_2+l)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}}. \tag{2.17}$$

The K matrix then assumes the striking form

$$K = \begin{bmatrix} 0 & -i\alpha_1 & & & & \\ i\alpha_1 & 0 & -i\alpha_2 & & \text{O} & \\ & i\alpha_2 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \text{O} & & \ddots & 0 & -i\alpha_{n-1} \\ & & & & i\alpha_{n-1} & 0 \end{bmatrix}. \tag{2.18}$$

2.3 Diagonalization

This is a particular Jacobi matrix, i.e., a matrix with all its nonvanishing elements along its principal diagonal and the two adjacent ones. These Jacobi matrices have many wonderful properties.[5] Here, further, K is an antisymmetric matrix. We get then the following

Lemma. The eigenvalues of K in each subspace $\mathcal{H}_{JM}$ are all different from each other.

We defer the proof to the Appendix B.

We can thus assert that the eigenvalues of K are effectively able to play their parts as degeneracy parameters. This shows also that the operators $(\mathbf{J}^1)^2$, $(\mathbf{J}^2)^2$, $(\mathbf{J}^3)^2$, $(\mathbf{J})^2$, J_0, and K constitute a *complete* set of commuting observables.

But K is antisymmetric, and, denoting by $Q(\lambda)$ the characteristic polynomial of K, we have

$$\begin{aligned} Q(\lambda) &= \det(K - \lambda\mathbb{1}) = \det(K^T - \lambda\mathbb{1}) \\ &= \det(-K - \lambda\mathbb{1}) \\ &= (-1)^n \det(K + \lambda\mathbb{1}) = (-1)^n Q(-\lambda), \end{aligned} \tag{2.19}$$

[5] F. R. Gantmacher and M. G. Krein, *Oscillatory Matrices and Small Oscillations of Mechanical Systems* (in Russian), (Fiz. Mat. Giz., Moscow, 1950).

where n is the degree of $P(\lambda)$, i.e., the dimension of the subspace $\mathcal{H}_{JM}$ considered. The characteristic polynomial of K is then of the same parity as its degree. Combining this with the preceding Lemma, we obtain the following

Theorem. The set of the eigenvalues of K in each subspace $\mathcal{H}_{JM}$ consists of pairs of opposite eigenvalues $(k_r, -k_r)$, these pairs being different from each other, plus at most a zero eigenvalue.

It seems in general impossible to compute explicitly the eigenvalues of K, the computation depending on the solution of high degree algebraic equations. This is the main drawback of our method. It can be seen that these eigenvalues generally are not rational functions of j_1, j_2, j_3. However, when the $\mathcal{H}_{JM}$ subspace is of low dimension, i.e., for the highest possible values of J, the computation is possible. We thus find the following eigenvalues for K:

if

$$\begin{aligned} &J = j_1 + j_2 + j_3, \qquad k = 0; \\ &J = j_1 + j_2 + j_3 - 1, \quad k = \pm[j_1j_2j_3(j_1+j_2+j_3)]^{\frac{1}{2}}; \\ &J = j_1 + j_2 + j_3 - 2, \\ &k = 0, \pm[4j_1j_2j_3(j_1+j_2+j_3) \\ &\qquad - (j_1+j_2+j_3)(j_1j_2+j_2j_3+j_3j_1) \\ &\qquad - 3j_1j_2j_3 + (j_1j_2+j_2j_3+j_3j_1)]^{\frac{1}{2}}. \end{aligned} \tag{2.20}$$

One obviously encounters the same difficulties when computing the eigenvectors of K.

From now on we denote by $|kJM\}$ the eigenvectors of K, $(\mathbf{J})^2$, J_0 in order to distinguish them from the eigenvectors $|j_{12}JM\rangle$ of $(\mathbf{J}_1 + \mathbf{J}_2)^2$, $(\mathbf{J})^2$ and J_0.

3. SYMMETRICAL COUPLING OF THREE IDENTICAL ANGULAR MOMENTA

3.1. K Eigenvectors and the Symmetric Group

The vector space $\mathcal{H}_j^{\otimes 3}$ of the representation is also a representation space for the symmetric group of degree three, S_3.[2]

To each permutation $\binom{1\ 2\ 3}{i_1\ i_2\ i_3}$ corresponds an operator $P_{i_1i_2i_3}$ acting on the basis vectors $|m_1m_2m_3\rangle$, which are eigenstates of J_0^1, J_0^2, and J_0^3, according to

$$P_{i_1i_2i_3}\ |m_1m_2m_3\rangle = |m_{i_1}m_{i_2}m_{i_3}\rangle. \tag{3.1}$$

These permutations induce automorphisms in the algebra of the angular momentum operators, according to

$$P_{i_1i_2i_3}\mathbf{J}^{\alpha}P^{-1}_{i_1i_2i_3} = \mathbf{J}^{i_\alpha}. \tag{3.2}$$

As a consequence we have

$$P_{i_1i_2i_3}\mathbf{J}P^{-1}_{i_1i_2i_3} = \mathbf{J}, \tag{3.3}$$

$$P_{i_1i_2i_3}KP^{-1}_{i_1i_2i_3} = \pm K, \tag{3.4}$$

when the permutation is even or odd, respectively.

The permutation group S_3 has two generators, for example the interchange permutation $P_{12} = \binom{123}{213}$ and the cyclic permutation $\mathcal{C} = \binom{123}{231}$; from now on, it will be sufficient to consider only the corresponding operators.

From (3.3) we see that the subspaces $\mathcal{H}_{JM}$ are invariant under the operations of S_3. Therefore, it is sufficient to consider the reduction of the S_3 representation in each of these subspaces.

But, from (3.4),

$$P_{12}KP_{12}^{-1} = -K, \qquad \mathcal{C}K\mathcal{C}^{-1} = K, \tag{3.5}$$

which proves that (a) if $k = 0$, the vector $|0JM\}$ is a basis for a one-dimensional (irreducible) representation of S_3; (b) if $k \neq 0$, the vectors $|kJM\}$ and $|-kJM\}$ for each value of $|k|$, form a basis for a two-dimensional representation of S_3.

Now there exist three irreducible representations of S_3: two are one-dimensional, the symmetrical one (which we note $\mathcal{S}$) and the antisymmetrical one ($\mathcal{A}$); one is two-dimensional, the "mixed" representation ($\mathcal{M}$).

We are now faced with two questions: (a) if $k = 0$, does the vector $|0JM\}$ belong to $\mathcal{S}$ or to $\mathcal{A}$? (b) if $k \neq 0$, do the vectors $|kJM\}$ and $|-kJM\}$ belong to an irreducible representation $\mathcal{M}$, or to a reducible one?

(a) To answer the first question is easy. Due to the very particular form of the matrix K in the basis $|j_{12}JM\rangle$ the vectors $|0JM\}$ have nonvanishing components only on those vectors $|j_{12}JM\rangle$, such that j_{12} has the same parity as the maximum (or minimum) j_{12} in the subspace $\mathcal{H}_{JM}$.

For a given J, the possible values of j_{12} are

$$J - j \le j_{12} \le 2j \qquad \text{if} \quad J \ge j,$$

$$J - j \le j_{12} \le j + J \quad \text{if} \quad J \le j.$$

Now

$$P_{12}\ |j_{12}JM\rangle = (-)^{2j-j_{12}}\ |j_{12}JM\rangle, \tag{3.6}$$

so that

$$P_{12}\ |0JM\} = |0JM\} \quad \text{if} \quad J \ge j,$$

and

$$P_{12}\ |0JM\} = (-)^{j+J}\ |0JM\} \qquad \text{if } J \le j \text{ and } j \text{ is an integer.} \tag{3.7}$$

[When $J \le j$ and both are half-integer, the $\mathcal{H}_{JM}$ subspaces have even dimension and there are no $|0JM\}$ vectors.]

Consequently, we know whether $|0JM\}$ is symmetrical or antisymmetrical. As is well known, the "scalar" $|000\}$ is symmetrical or antisymmetrical according as j is even or odd.

(b) If $k \neq 0$, after (3.5) one sees that $P_{12}\ |kJM\} \approx |-kJM\}$, so that P_{12} is traceless.

According to the table of the primitive characters of S_3 (Table I), the two-dimensional representations

TABLE I. Primitive characters of S_3.

	rep.		
class	$\mathcal{S}$	$\mathcal{M}$	$\mathcal{A}$
1^3	1	2	1
12	1	0	−1
3	1	−1	1

which have been considered, can only be $\mathcal{M}$ if it is irreducible, or $\mathcal{S} + \mathcal{A}$ if it is reducible. The eigenvalues k cannot be calculated explicitly, and thus, we are not able to say which is, in general, the situation for a given value k. The only information we get is the number of representations of type $\mathcal{M}$ or of type $\mathcal{S}+\mathcal{A}$, to be found in a given subspace $\mathcal{H}_{JM}$.

3.2. Reduction of the Symmetric Group Representation

Let us calculate the characters of the S_3 representation which is induced in a subspace $\mathcal{H}_{JM}$.

TABLE II. Characters of the representations of S_3 in the subspaces $\mathcal{H}_{JM}$ for $J \geq j$.

class	J: $3j$	$3j-1$	$3j-2$	$3j-3$	$3j-4$	$3j-5$	$3j-6$	...
1^3	1	2	3	4	5	6	7	...
12	1	0	1	0	1	0	1	...
3	1	−1	0	1	−1	0	1	...

It is clear that

$$\chi_{JM}(\mathbb{1}) = \text{dim of the representation}$$

$$= \begin{matrix} 3j - J + 1 & \text{for} & J \geq j, \\ 2J + 1 & \text{for} & J \leq j. \end{matrix} \tag{3.8}$$

On the other hand, from (3.6) and (3.7),

$$\chi_{JM}(P_{12}) = \sum_{j_{12}} (-)^{2j - j_{12}}$$

$$= \begin{matrix} +1 & \text{for } J - 3j \text{ even}, & \text{if} \quad J \geq j; \\ 0 & \text{for } J - 3j \text{ odd}, & \\ (-)^{J+j} & \text{for } j \text{ integer}, & \text{if} \quad J \leq j. \\ 0 & \text{for } j \text{ half integer}, & \end{matrix} \tag{3.9}$$

Finally, as concerns the cyclic permutations class, the simplest way to evaluate the character is to remark that if $\mathcal{K}_M$ is a subspace of given M (whatever J is), then

$$\mathcal{K}_J = J_- \mathcal{K}_{J+1} + \mathcal{H}_{JJ},$$

with obvious notations.

Observing that J_- commutes with the permutations and that χ_{JM} is independent of M, we see that

$$\kappa_J = \kappa_{J+1} + \chi_{JM},$$

where κ_J is the character of the considered permutation in the S_3 representation in $\mathcal{K}_J$.

We can use this method for all permutations, and especially for the cyclic permutations, we get

$$\kappa_J = \sum_{m_1+m_2+m_3=J} \delta_{m_1m_2}\delta_{m_2m_3}\delta_{m_3m_1} = \delta_{J,3l},$$

where l and j (and J) are integer or half-integer.

And finally,

$$\chi_{JM}(\mathfrak{C}) = \delta_{J,3l} - \delta_{J+1,3l'}. \tag{3.10}$$

So we have the complete table of characters. We give it for $J \geq j$ (Table II).

It is then easy, using the ordinary orthogonality relations for the primitive characters, to obtain the complete reduction of the S_3 representation in each subspace $\mathcal{H}_{JM}$. Table III gives the results of this reduction.

We recover the symmetry of the vectors of type $|0JM\}$, which we already obtained in (3.7), and we recover also the fact that, except for the corresponding $\mathcal{S}$ or $\mathcal{A}$ representation, we only have representations of type $\mathfrak{M}$ or $\mathcal{S} + \mathcal{A}$.

These results provide useful information for some $6j$ coefficients connected with the matrix elements of the S_3 representations in the subspace $\mathcal{H}_{JM}$ (see Appendix C).

Let us remark finally that the change of basis

$$|k_\pm JM\} = (1/\sqrt{2})[|kJM\} \pm |-kJM\}]$$

enables one to (a) reduce explicitly the representations of type $\mathcal{S} + \mathcal{A}$ (i.e., the vectors $|k_\pm JM\}$ are completely symmetrical or completely antisymmetrical); (b) to put into real form the irreducible representations of type $\mathfrak{M}$.

4. EXAMPLES

In the following tables, we give the components of the vectors $|kJM\}$ in the basis $|m_1m_2m_3\rangle$ for the cases

TABLE III. Irreducible representations content of the S_3 representations in the subspaces $\mathcal{H}_{JM}$.

$J \geq j$ (l an integer)	
$J = 3j - 6l$	$\mathcal{S} + l(\mathcal{S} + \mathcal{A}) + 2l\mathfrak{M}$
$J = 3j - 6l - 1$	$l(\mathcal{S} + \mathcal{A}) + (2l + 1)\mathfrak{M}$
$J = 3j - 6l - 2$	$\mathcal{S} + l(\mathcal{S} + \mathcal{A}) + (2l + 1)\mathfrak{M}$
$J = 3j - 6l - 3$	$(l + 1)(\mathcal{S} + \mathcal{A}) + (2l + 1)\mathfrak{M}$
$J = 3j - 6l - 4$	$\mathcal{S} + l(\mathcal{S} + \mathcal{A}) + (2l + 2)\mathfrak{M}$
$J = 3j - 6l - 5$	$(l + 1)(\mathcal{S} + \mathcal{A}) + (2l + 2)\mathfrak{M}$
$J \leq j$, j integer; [l an integer, and $\mathcal{R} = \mathcal{S}$ (resp. $\mathcal{A}$) if j and J have the same parity (resp. opposite parity)]	
$J = 3l$	$\mathcal{R} + l(\mathcal{S} + \mathcal{A}) + 2l\mathfrak{M}$
$J = 3l + 1$	$\mathcal{R} + l(\mathcal{S} + \mathcal{A}) + (2l + 1)\mathfrak{M}$
$J = 3l + 2$	$\mathcal{R} + l(\mathcal{S} + \mathcal{A}) + (2l + 2)\mathfrak{M}$
$J \leq j$, j half-integer; (l half-integer)	
$J = 3l$	$(l + \frac{1}{2})(\mathcal{S} + \mathcal{A}) + 2l\mathfrak{M}$
$J = 3l + 1$	$(l + \frac{1}{2})(\mathcal{S} + \mathcal{A}) + (2l + 1)\mathfrak{M}$
$J = 3l + 2$	$(l + \frac{1}{2})(\mathcal{S} + \mathcal{A}) + (2l + 2)\mathfrak{M}$

TABLE IV. Case $j_1 = j_2 = j_3 = \frac{1}{2}$, where $\omega = \exp(\frac{2}{3}i\pi)$.

	$\vert +++\rangle$	$\vert +-+\rangle$	$\vert ++-\rangle$	$\vert -++\rangle$
$\vert 0, \frac{3}{2}, \frac{3}{2}\}$	1			
$\vert 0, \frac{3}{2}, \frac{1}{2}\}$		$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
$\vert +\sqrt{3}, \frac{1}{2}, \frac{1}{2}\}$		$\omega^2/\sqrt{3}$	$1/\sqrt{3}$	$\omega/\sqrt{3}$
$\vert -\sqrt{3}, \frac{1}{2}, \frac{1}{2}\}$		$\omega/\sqrt{3}$	$1/\sqrt{3}$	$\omega^2\ \sqrt{3}$

$$j_1 = j_2 = j_3 = \tfrac{1}{2} \qquad \text{(Table IV)},$$

$$j_1 = j_2 = j_3 = 1 \qquad \text{(Table V)}.$$

Observe that for these values of j, the dimensions of the subspaces $\mathcal{H}_{JM}$ are always lower than three, so that the reducible representations $\mathcal{S} + \mathcal{A}$ do not appear. This only occurs for $j \geq \frac{3}{2}$.

We obtain the coefficients corresponding to $M < 0$ by noticing that

$$\{kJ\ -M \mid -m_1\ -m_2\ -m_3\rangle = (-)^{j_1+j_2+j_3-J}\{kJM \mid m_1m_2m_3\rangle.$$

In this particular case, we notice that the states of the type

$$|\sqrt{3}\pm, \tfrac{1}{2}, m\} = (1/\sqrt{2})[|\sqrt{3}, \tfrac{1}{2}, m\} \pm |-\sqrt{3}, \tfrac{1}{2}, m\}]$$

are nothing else but the basis states which correspond to the usual coupling $|j_{12}, \frac{1}{2}, m\rangle$ for $j_{12} = 1, 0$. This comes from the following fact: the subspaces $J = \frac{1}{2}$, $M = \pm\frac{1}{2}$ are two-dimensional and, in each of these subspaces, the induced S_3 representation is irreducible, which is already true with the usual way of coupling.

5. CONCLUSION

We have shown that a new coupling scheme for three angular momenta is obtained by diagonalizing the K operator. In this scheme, the individual angular momenta are treated alike. As a consequence, when these are equal, the eigenvectors of K and the total angular momentum belong to subspaces which are irreducible with respect to the representations of the symmetric group S_3. These eigenvectors have thus simple symmetry properties. These should be useful when one deals with spin or isospin states for three bosons[6] or three fermions.

(This work was practically completed when we learned that Dr. Chakrabarti had simultaneously obtained similar results.[7])

[6] Such three-pion $I = 1$ isospin states have been used for instance by A. J. Dragt, J. Math. Phys. 6, 533 (1965).

[7] A. Chakrabarti, Ann. Inst. Henri Poincaré 1, 301 (1964).

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APPENDIX A

We want to evaluate the matrix element

$$\langle j'_{12}JM|\ (\mathbf{J}_2 + \mathbf{J}_3)^2\ |j_{12}JM\rangle. \tag{A1}$$

This reduces immediately to

$$[j_2(j_2+1) + j_3(j_3+1)]\delta_{j'_{12},j_{12}} + 2\,\langle j'_{12}JM|\ \mathbf{J}_2\cdot\mathbf{J}_3\ |j_{12}JM\rangle. \tag{A2}$$

But $\mathbf{J}_2\cdot\mathbf{J}_3$ is the scalar product of two irreducible tensor operators operating respectively on the (1-2) and (3) parts of the system. Standard methods[4] show that

$$\langle j'_{12}JM|\ \mathbf{J}_2\cdot\mathbf{J}_3\ |j_{12}JM\rangle = (-1)^{j_{12}+j_3+J}\begin{Bmatrix} J & j_3 & j'_{12} \\ 1 & j_{12} & j_3 \end{Bmatrix} \times (j'_{12}\ ||J_2||\ j_{12})(j_3\ ||J_3||\ j_3). \tag{A3}$$

The reduced matrix element $(j'_{12}\ ||J_2||\ j_{12})$ can be evaluated by similar methods:

$$(j'_{12}\ ||J_2||\ j_{12}) = (-1)^{j_1+j_2+j'_{12}+1}[(2j_{12}+1)(2j'_{12}+1)]^{\frac{1}{2}} \times \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ 1 & j_{12} & j_2 \end{Bmatrix}(j_2\ ||J_2||\ j_2). \tag{A4}$$

Finally, since

$$(j\ ||J||\ j) = [j(j+1)(2j+1)]^{\frac{1}{2}}, \tag{A5}$$

we obtain

$$\begin{aligned}\langle j'_{12}JM|\ \mathbf{J}_2\cdot\mathbf{J}_3\ |j_{12}JM\rangle &= (-1)^{j_1+j_2+j_3+j_{12}+j'_{12}+J+1} \\ &\times [j_2(j_2+1)(2j_2+1)j_3(j_3+1)(2j_3+1)]^{\frac{1}{2}} \\ &\times [(2j_{12}+1)(2j'_{12}+1)]^{\frac{1}{2}} \\ &\times \begin{Bmatrix} J & j_3 & j'_{12} \\ 1 & j_{12} & j_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j'_{12} \\ 1 & j_{12} & j_2 \end{Bmatrix}.\end{aligned} \tag{A6}$$

This matrix element is nonvanishing if and only if the selection rule

$$|j'_{12} - j_{12}| \leq 1 \tag{A7}$$

is satisfied. We could have evaluated the matrix elements of $(\mathbf{J}_2 + \mathbf{J}_3)^2$ in the basis $|j_{23}JM\rangle$, and

TABLE V. Case $j_1 = j_2 = j_3 = 1$.

	$\lvert m_1 m_2 m_3\rangle$																
$\lvert KJM\}$	$\lvert 111\rangle$	$\lvert 110\rangle$	$\lvert 101\rangle$	$\lvert 011\rangle$	$\lvert 11{-1}\rangle$	$\lvert {-1}11\rangle$	$\lvert 1{-1}1\rangle$	$\lvert 001\rangle$	$\lvert 100\rangle$	$\lvert 010\rangle$	$\lvert 1{-1}0\rangle$	$\lvert 01{-1}\rangle$	$\lvert {-1}01\rangle$	$\lvert {-1}10\rangle$	$\lvert 0{-1}1\rangle$	$\lvert 10{-1}\rangle$	$\lvert 000\rangle$
$\lvert 0, 3, 3\}$	1																
$\lvert 0, 3, 2\}$		$\frac{1}{[3]^{1/2}}$	$\frac{1}{[3]^{1/2}}$	$\frac{1}{[3]^{1/2}}$													
$\lvert +[3]^{1/2}, 2, 2\}$		$\frac{1}{[3]^{1/2}}$	$\frac{\omega^2}{[3]^{1/2}}$	$\frac{\omega}{[3]^{1/2}}$													
$\lvert -[3]^{1/2}, 2, 2\}$		$\frac{1}{[3]^{1/2}}$	$\frac{\omega}{[3]^{1/2}}$	$\frac{\omega^2}{[3]^{1/2}}$													
$\lvert 0, 3, 1\}$					$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{2}{[15]^{1/2}}$	$\frac{2}{[15]^{1/2}}$	$\frac{2}{[15]^{1/2}}$							
$\lvert +[3]^{1/2}, 2, 1\}$					$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$							
$\lvert -[3]^{1/2}, 2, 1\}$					$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$							
$\lvert +[3]^{1/2}, 1, 1\}$					$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$	$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$							
$\lvert -[3]^{1/2}, 1, 1\}$					$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$	$-\frac{1}{[6]^{1/2}}$	$-\frac{\omega^2}{[6]^{1/2}}$	$-\frac{\omega}{[6]^{1/2}}$							
$\lvert 0, 1, 1\}$					$\frac{2}{[15]^{1/2}}$	$\frac{2}{[15]^{1/2}}$	$\frac{2}{[15]^{1/2}}$	$-\frac{1}{[15]^{1/2}}$	$-\frac{1}{[15]^{1/2}}$	$-\frac{1}{[15]^{1/2}}$							
$\lvert 0, 3, 0\}$											$\frac{1}{[10]^{1/2}}$	$\frac{1}{[10]^{1/2}}$	$\frac{1}{[10]^{1/2}}$	$\frac{1}{[10]^{1/2}}$	$\frac{1}{[10]^{1/2}}$	$\frac{1}{[10]^{1/2}}$	$\frac{2}{[10]^{1/2}}$
$\lvert +[3]^{1/2}, 2, 0\}$											$-\frac{i}{[6]^{1/2}}$	$-\frac{i\omega}{[6]^{1/2}}$	$-\frac{i\omega^2}{[6]^{1/2}}$	$\frac{i}{[6]^{1/2}}$	$\frac{i\omega}{[6]^{1/2}}$	$\frac{i\omega^2}{[6]^{1/2}}$	0
$\lvert -[3]^{1/2}, 2, 0\}$											$\frac{i}{[6]^{1/2}}$	$\frac{i\omega^2}{[6]^{1/2}}$	$\frac{i\omega}{[6]^{1/2}}$	$-\frac{i}{[6]^{1/2}}$	$-\frac{i\omega^2}{[6]^{1/2}}$	$-\frac{i\omega}{[6]^{1/2}}$	0
$\lvert +[3]^{1/2}, 1, 0\}$											$\frac{1}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$	0
$\lvert -[3]^{1/2}, 1, 0\}$											$\frac{1}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$\frac{\omega^2}{[6]^{1/2}}$	$\frac{\omega}{[6]^{1/2}}$	0
$\lvert 0, 1, 0\}$											$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$\frac{1}{[15]^{1/2}}$	$-\frac{3}{[15]^{1/2}}$
$\lvert 0, 0, 0\}$											$\frac{1}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$\frac{1}{[6]^{1/2}}$	$-\frac{1}{[6]^{1/2}}$	$-\frac{1}{[6]^{1/2}}$	$-\frac{1}{[6]^{1/2}}$	0

then transform back to the basis $|j_{12}JM\rangle$. Such a method gives the formula

$$\langle j'_{12}JM|\,(\mathbf{J}_2+\mathbf{J}_3)^2\,|j_{12}JM\rangle$$
$$= [(2j_{12}+1)(2j'_{12}+1)]^{\frac{1}{2}} \sum_{j_{23}} j_{23}(j_{23}+1)(2j_{23}+1)$$
$$\times \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & J & j_{23} \end{Bmatrix}. \quad \text{(A8)}$$

The preceding computations furnish us with the value of this sum.

APPENDIX B

Lemma. In each subspace $\mathcal{H}_{JM}$, the eigenvalues of K are different from each other

Proof.[5] It depends only on K being a Jacobi matrix. Let $P_j(\lambda)$, $j = 1, \cdots, n$, be the principal minors of the matrix $-K + \lambda\mathbb{1}$,

$$P_j(\lambda) = \det \begin{bmatrix} \lambda & i\alpha_1 & & & & \\ -i\alpha_1 & \lambda & i\alpha_2 & & \mathbf{O} & \\ & -i\alpha_2 & \lambda & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & \mathbf{O} & & \cdot & \lambda & i\alpha_{j-1} \\ & & & & -i\alpha_{j-1} & \lambda \end{bmatrix}. \quad \text{(B1)}$$

We get at once the following recursion formula:

$$P_j(\lambda) = \lambda P_{j-1}(\lambda) - \alpha_{j-1}^2 P_{j-2}(\lambda) \quad \text{(B2)}$$

with

$$P_0 = 1, \qquad P_1 = \lambda \quad \text{(B2')}$$

so that

$$P_2 = \lambda^2 - \alpha_1^2, \text{ etc.}$$

Two successive polynomials P_{j-1} and P_j cannot have a common root. If they had, after (B2) and since $\alpha_{j-1} \neq 0$ (See 2.17), this common root would also be a root of $P_{j-2} \cdots$, then of P_0, which is impossible. We show now that the roots $\lambda_k^{(j)}$ of P_j strictly separate the roots $\lambda_k^{(j+1)}$ of P_{j+1}, i.e.,

$$-\infty < \lambda_1^{(j+1)} < \lambda_1^{(j)} < \lambda_2^{(j+1)}$$
$$< \lambda_2^{(j)} \cdots < \lambda_j^{(j+1)} < \lambda_j^{(j)} < \lambda_{j+1}^{(j+1)} < +\infty. \quad \text{(B3)}$$

This is obviously true for P_1 and P_2. Suppose it to be true for P_j and P_{j-1},

$$-\infty < \lambda_1^{(j)} < \lambda_1^{(j-1)}$$
$$< \lambda_1^{(j)} \cdots < \lambda_{j-1}^{(j)} < \lambda_j^{(j-1)} < \lambda_j^{(j)} < +\infty. \quad \text{(B3')}$$

But, after (B2),

$$P_{j+1}(\lambda) = \lambda(\lambda - \lambda_1^{(j)}) \cdots (\lambda - \lambda_j^{(j)})$$
$$- \alpha_j^2(\lambda - \lambda_1^{(j-1)}) \cdots (\lambda - \lambda_{j-1}^{(j-1)}). \quad \text{(B4)}$$

Then

$$P_{j+1}(\lambda_k^{(j)}) = -\alpha_j^2(\lambda_k^{(j)} - \lambda_1^{(j-1)}) \cdots (\lambda_k^{(j)} - \lambda_{j-1}^{(j-1)}). \quad \text{(B5)}$$

After the hypothesis (B.3′), $P_{j+1}(\lambda_k^{(j)})$ is of sign $(-1)^{j+k+1}$. $P_{j+1}(\lambda)$ modify its sign between $\lambda = \lambda_k^{(j)}$ and $\lambda = \lambda_{k+1}^{(j)}$. There is thus at least one root of P_{j+1} between two successive roots of P_j. But, on the other hand, for λ positive great enough $P_{j+1}(\lambda)$ is certainly positive although $P_{j+1}(\lambda_j^{(j)}) < 0$. So, there is a root of P_{j+1} greater than $\lambda_j^{(j)}$. In the same way, one shows there is a root of P_{j+1} smaller than $\lambda_1^{(j)}$. But P_{j+1} cannot have more than $(j + 1)$ roots. We have then proved the inequalities (B.3). In particular, all the roots of any given P_j are different from each other. This is true in particular for $j = n$, that is for the characteristic polynomial of $(-k+\lambda\mathbb{1})$.

APPENDIX C

We have seen that, when dealing with three equal angular momenta j, each of the subspaces $\mathcal{H}_{JM}$ is a representation space for a representation of the symmetric group S_3. It is easy to derive explicitly the matrices corresponding to each element of S_3 in this representation. Their matrix elements are 6 j-symbols. We introduce the matrices L, M, N with the following definitions:

$$L_{kl} = (-1)^k \delta_{kl}, \quad \text{(C1)}$$
$$M_{kl} = [(2k+1)(2l+1)]^{\frac{1}{2}} \left\{\begin{smallmatrix} j & j & k \\ j & J & l \end{smallmatrix}\right\}, \quad \text{(C2)}$$
$$N_{kl} = (-1)^k[(2k+1)(2l+1)]^{\frac{1}{2}} \left\{\begin{smallmatrix} j & j & k \\ j & J & l \end{smallmatrix}\right\}. \quad \text{(C3)}$$

These matrices are orthogonal and possess some remarkable properties (due to those of the 6 j-symbols):

$$MM^T = \mathbb{1}, \quad M = M^T,$$
$$\text{or} \quad M = M^T = M^{-1}, \quad M^2 = \mathbb{1}, \quad \text{(C4)}$$
$$NN^T = \mathbb{1}, \quad N^2 = N^T,$$
$$\text{or} \quad N^2 = N^T = N^{-1}, \quad N^3 = \mathbb{1}, \quad \text{(C5)}$$

and

$$MN = NL, \quad N = LM, \quad NM = L, \text{ etc} \cdots. \quad \text{(C6)}$$

The elements $(1, \mathcal{P}_{12}, \mathcal{P}_{23}, \mathcal{P}_{31}, \mathcal{C}, \mathcal{C}^2)$ of S_3 are, respectively, represented by the matrices $(\mathbb{1}, L, NL, M, N, N^2)$. We can use the relations (C4), (C5), (C6), to verify the group law. But since we have reduced the above representation into irreducible components, we can derive some results about the matrices M and N. At first, since M belongs to the conjugation class of the transpositions as do L, M and L have the same eigenvalues. Actually, the relation

$$MN = NL \quad \text{or} \quad \sum_j M_{ij}N_{jk} = (-1)^k N_{ik} \tag{C7}$$

shows that M has eigenvalues $(-1)^k$ with the kth column of N as eigenvector. This is a known result.[8]

What is more interesting is the computation of the eigenvalues of N. Due to the fact that N is real with $N^3 = \mathbb{1}$, these eigenvalues consist of 1 with multiplicity a, ω $(= e^{\frac{2}{3}i\pi})$ and ω^2 with the same multiplicity b. Naturally,

$$a + 2b = \dim N = \begin{matrix} 3j - J + 1 & \text{for} \quad J \geq j, \\ 2J + 1 & \text{for} \quad J \leq j. \end{matrix} \tag{C8}$$

But on the other hand we have already evaluated the character of the class to which N belongs. [See (3.10)]. Thus

$$\operatorname{Tr} N = a - b = \begin{cases} +1 & \\ 0 & \text{for } J \text{ of the form } 3l,\ 3l+1,\ 3l+2 \text{ resp.} \\ -1 & \end{cases} \tag{C9}$$

Combining (C8) and (C9), one gets a and b.

Other properties can be obtained in the same way.

[8] M. E. Rose and C. N. Yang, J. Math. Phys. 3, 106 (1962).

On the Possibility of Relating Internal Symmetries and Lorentz Invariance*

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In this note we investigate the possibility that the inhomogeneous Lorentz group is only a subgroup of a larger Lie group G of symmetries for strong interaction physics. The discussion is restricted to the Lie algebra $\mathfrak{L}$ of G. We make the assumption that the remaining generators A_i of G commute with the generators of translations P^λ which build the ideal $\mathfrak{J}$ in $\mathfrak{L}$. It is then shown that the A_i generate an ideal in $\mathfrak{L}$ modulo $\mathfrak{J}$. If this ideal is semi-simple then $\mathfrak{L}$ breaks up in a direct sum $\mathfrak{L} = \mathcal{P} \oplus \mathfrak{A}$, where $\mathcal{P}$ is isomorphic with the Lie algebra of the inhomogeneous Lorentz group and $\mathfrak{A}$ is semisimple.

1. INTRODUCTION

It is interesting to discuss the various possibilities of mixing "internal symmetries" in strong interaction physics with Lorentz invariance. However, this cannot be done in an arbitrary way as was recently shown by McGlinn.[1] We take here a point of view closely related to McGlinn's, by assuming that the inhomogeneous Lorentz group is only a subgroup of a larger Lie group G. This is not the only possibility as pointed out by Michel[2] who assumes the inhomogeneous Lorentz group to be only a factor group of G. However, we do not discuss here these other ways of interrelating internal symmetries and Lorentz invariance. In fact we restrict our discussion to the Lie algebra of G, call it $\mathfrak{L}$. Our main hypothesis will be that it is possible to choose a basis of this algebra in such a way that:

(i) The generators of the usual translations P^λ, $0 \leq \lambda \leq 3$ and homogeneous Lorentz transformations $M_{\mu\nu} = -M_{\nu\mu}$, $\mu, \nu = 0, 1, 2, 3$ have their usual commutation relations (this being another way of describing the fact that the inhomogeneous Lorentz group is a subgroup of G);

(ii) The other generators $A_i 1 \leq i \leq n$ commute with P^λ. This assures that any irreducible representation of our group G will be characterized by a common mass so that we do not expect the relation of internal symmetries to Lorentz invariance to account for the actual mass splitting within the same supermultiplet. Had we only assumed $[A_i, P^0] = 0$, then Lorentz covariance would require it to be true in any frame so that $[A_i, P^\lambda] = 0$ whatever the index λ. Now we will show the following:

Theorem: Under assumptions (i) and (ii), the P^λ generate an ideal $\mathfrak{J}$ in $\mathfrak{L}$. The images of the A_i under the application $\mathfrak{L} \to \mathfrak{L}/\mathfrak{J} = \mathfrak{L}'$ generate an ideal $\mathfrak{A}'$ in $\mathfrak{L}'$. If we assume that this algebra $\mathfrak{A}'$ is semisimple, then $\mathfrak{L}$ is equal to a direct sum of Lie algebras $\mathfrak{L} = \mathfrak{A} \oplus \mathcal{P}$, where $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$ and $\mathcal{P}$ is isomorphic to the usual Lie algebra of the inhomogeneous Lorentz group.

This result is in some sense parallel to the one of McGlinn's. The essential difference being here that in order to split $\mathfrak{L}$ into this direct sum one may have to change the labels of the operators. Before proving the theorem in Sec. 4, we recall in Sec. 2 some definitions about Lie algebras and devote Sec. 3 to the proof of two lemmas which will clear the way. Section 5 gives some comments on our result.

For convenience we choose all our Lie algebras to be on the real numbers. In this way P^λ and $M_{\mu\nu}$ have the commutation relations of $\partial/\partial x_\lambda$ and $x_\mu \partial/\partial x^\nu - x_\nu \partial/\partial x^\nu$, respectively. The Greek indices will always run from 0 to 3; $g_{\mu\nu} = g^{\mu\nu}$ is the usual diagonal real Lorentz metric. We use freely of $g_{\mu\nu}$ to raise or lower indices.[3]

2. DEFINITIONS

To make our language precise, we recall briefly some facts about Lie algebras. For details one can refer for instance to Helgasson's book.[4] By Lie algebra $\mathfrak{L}$ over R, the real numbers, we mean a finite dimensional vector space over R with an internal bilinear product satisfying $[x, x] = 0$ and the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

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† On leave from Service de Physique Théorique, C. E. N., Saclay, B. P. Nr. 2, Gif-sur-Yvette (Seine-et-Oise), France.

[1] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964).

[2] L. Michel, Phys. Rev. **137**, B405 (1965).

[3] Whenever two indices, one upper and one lower, appear in the same formula and are equal, they are meant to be summed over; the summing for Greek indices to be from 0 to 3, the summing for Latin indices to be from 1 to n.

[4] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York and London, 1962).

A subalgebra of $\mathfrak{L}$ is a vector subspace closed under the product. An ideal is a true vector subspace $\mathfrak{I} \subset \mathfrak{L}$ such that for any $l \in \mathfrak{L}$ and $i \in \mathfrak{I}$, $[l, i]$ belongs to $\mathfrak{I}$. The factor algebra $\mathfrak{L}/\mathfrak{I}$ is the factor vector space with the inner product inherited from $\mathfrak{L}$. We shall say that a Lie algebra $\mathfrak{L}$ is a direct sum of two ideals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ and write $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ if $[l_1, l_2] = 0$, $[l_1', l_1''] = \in \mathfrak{L}_1$, $[l_2', l_2''] \in \mathfrak{L}_2$, $l_1, l_1', l_1'' \in \mathfrak{L}_1$, $l_2, l_2', l_2'' \in \mathfrak{L}_2$ and if any element l in $\mathfrak{L}$ can be uniquely written as $l = l_1 + l_2$. Mathematicians often call $\mathfrak{L}$ the direct product of $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

Given $l \in \mathfrak{L}$ the linear correspondence $x \to [l, x] = \text{adj } (l)\cdot x$ has the property adj l' adj l'' − adj l'' adj l' = adj $[l', l'']$ and defines the adjoint representation of $\mathfrak{L}$: adj ($\mathfrak{L}$). If the symmetric bilinear form $K(l_1', l'') = \text{Trace adj } l' \text{ adj } l''$ is nondegenerate, the Lie algebra is called semisimple. In this case $\mathfrak{L}$ has no Abelian ideal except zero, and its center is also zero. Finally, by derivation on a Lie algebra one means a linear correspondence $x \to Dx$ with the property

$$D[x, y] = [Dx, y] + [x, Dy].$$

These derivations themselves build up a Lie algebra $\mathfrak{D}(\mathfrak{L})$, with $[D', D''] = D'D'' - D''D'$. It is easy to see that the adj ($\mathfrak{L}$) is an ideal in $\mathfrak{D}(\mathfrak{L})$ (in this respect adj l is also called an inner derivation of $\mathfrak{L}$) with

$$[D, \text{adj } l] = \text{adj } D(l).$$

Now the main result we shall use is the following.

Theorem: For a semisimple Lie algebra every derivation is an inner derivation. In other words, $\mathfrak{D}(\mathfrak{L}) = \text{adj } (\mathfrak{L})$. The proof of this theorem is given on p. 122 of Ref. 4.

3. TWO LEMMAS

In this section we prove the following two lemmas:

Lemma 1: Suppose that we have a Lie algebra over R and that, in terms of a basis P^λ, $N_{\mu\nu}$, the following commutation relations hold:

$$\begin{aligned}
&[P^\lambda, P^\rho] = 0, \qquad [P^\lambda, N_{\mu\nu}] = \delta_\mu^\lambda P_\nu - \delta_\nu^\lambda P_\mu, \\
&[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}] = g_{\mu_1\nu_2} N_{\nu_1\mu_2} + g_{\nu_1\mu_2} N_{\mu_1\nu_2} \\
&\qquad - g_{\mu_1\mu_2} N_{\nu_1\nu_2} - g_{\nu_1\nu_2} N_{\mu_1\mu_2} \\
&\qquad + a_{\mu_1\nu_1,\mu_2\nu_2,\lambda} P^\lambda, \\
&\qquad N_{\mu\nu} = -N_{\nu\mu},
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
a_{\mu_1\nu_1,\mu_2\nu_2,\lambda} &= -a_{\nu_1\mu_1,\mu_2\nu_2,\lambda} \\
&= -a_{\mu_1\nu_1,\nu_2\mu_2,\lambda} = -a_{\mu_2\nu_2,\mu_1\nu_1,\lambda}.
\end{aligned} \tag{2}$$

Then the Lie algebra just described is necessarily isomorphic to the one of the inhomogeneous Lorentz group.

Proof of Lemma 1[5]*:* The statement of the lemma can be rephrased in the following way. There exists a set of (real) coefficients $f_{\mu\nu,\lambda}$ such that putting $L_{\mu\nu}$ equal to $N_{\mu\nu} - f_{\mu\nu,\lambda}P^\lambda$, the $L_{\mu\nu}$ and P^λ generate the same Lie algebra as before; however, these new operators have among them the usual commutation relations of the generators of homogeneous Lorentz transformations and translations, respectively, that is, this change of basis has the virtue of suppressing the "a" coefficients in the commutation relations (1) when they are written in terms of P^λ and $L_{\mu\nu}$.

We have to make sure that our Lie algebra satisfies the Jacobi identities:

$$\sum_{\text{circular permutation of } 1,2,3} [N_{\mu_1\nu_1}, [N_{\mu_2\nu_2}, N_{\mu_3\nu_3}]] = 0.$$

This gives us a set of linear relations among the coefficients $a_{\mu_1\nu_1,\mu_2\nu_2,\lambda}$:

$$\begin{aligned}
\sum_{\substack{\text{circular permutation} \\ \text{of } 1,2,3}} &(g_{\mu_2\nu_3} a_{\mu_1\nu_1,\nu_2\mu_3,\lambda} + g_{\nu_2\mu_3} a_{\mu_1\nu_1,\mu_2\nu_3,\lambda} \\
&- g_{\nu_2\nu_3} a_{\mu_1\nu_1,\mu_2\mu_3,\lambda} - g_{\mu_2\mu_3} a_{\mu_1\nu_1,\nu_2\nu_3,\lambda} \\
&+ g_{\mu_1\lambda} a_{\mu_2\nu_2,\mu_3\nu_3,\nu_1} - g_{\nu_1\lambda} a_{\mu_2\nu_2,\mu_3\nu_3,\mu_1}) = 0.
\end{aligned} \tag{3}$$

Our task now is to find the most general solution of these equations. Let α, β, γ, δ stand for 0, 1, 2, 3 in any order. Because of the symmetry properties of the a's, we have only to compute the following quantities:

$$a_{\alpha\beta,\alpha\gamma,\delta}; \quad a_{\alpha\beta,\gamma\delta,\delta}; \quad a_{\alpha\beta,\alpha\gamma,\alpha}; \quad a_{\alpha\beta,\beta\gamma,\gamma}.$$

The value of all $a_{\mu_1\nu_1,\mu_2\nu_2,\lambda}$ is then related to the preceding ones using (2). Let us first write (3) for $\mu_1 = \alpha$, $\nu_1 = \gamma$, $\mu_2 = \beta$, $\nu_2 = \delta$, $\mu_3 = \alpha$, $\nu_3 = \beta$, and $\lambda = \alpha$:

$$\begin{aligned}
g_{\alpha\alpha}(a_{\beta\delta,\alpha\beta,\gamma} + a_{\alpha\gamma,\beta\delta,\beta} - a_{\beta\delta,\beta\gamma,\alpha}) \\
+ g_{\beta\beta} a_{\alpha\gamma,\delta\alpha,\alpha} = 0.
\end{aligned} \tag{4}$$

Now we put in (3) $\mu_1 = \alpha$, $\nu_1 = \beta$, $\mu_2 = \alpha$, $\nu_2 = \gamma$, $\mu_3 = \alpha$, $\nu_3 = \delta$, $\lambda = \beta$:

$$\begin{aligned}
(g_{\alpha\alpha} a_{\alpha\beta,\gamma\delta,\beta} + a_{\alpha\gamma,\delta\beta,\beta} + a_{\alpha\delta,\beta\gamma,\beta}) \\
+ g_{\beta\beta} a_{\alpha\gamma,\alpha\delta,\alpha} = 0.
\end{aligned} \tag{5}$$

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[5] In spite of the simplicity of the lemma, the author was unable to find a simpler proof than this rather lengthy one.

Let us add (4) and (5):

$$a_{\beta\delta,\alpha\beta,\gamma}+a_{\delta\alpha,\gamma\beta,\beta}=a_{\beta\gamma,\delta\beta,\alpha}+a_{\gamma\delta,\alpha\beta,\beta}. \quad (6)$$

Call

$$\varphi^{\beta}{}_{\delta\alpha\gamma}=a_{\beta\delta,\alpha\beta,}+a_{\delta\alpha,\gamma\beta,\beta}.$$

Obviously,

$$\varphi^{\beta}{}_{\delta\alpha\gamma}=-\varphi^{\beta}{}_{\alpha\delta\gamma}. \quad (7')$$

Equation (6) when translated in terms of φ reads

$$\varphi^{\beta}{}_{\delta\alpha\gamma}=\varphi^{\beta}{}_{\gamma\delta\alpha}. \quad (7'')$$

Hence, $\varphi^{\alpha}{}_{\beta\gamma\delta}$ is invariant by a circular permutation of $\beta\gamma\delta$ and changes sign under an odd permutation. We will get a new equation for φ if we write (3) with $\mu_1=\alpha$, $\nu_1=\beta$, $\mu_2=\alpha$, $\nu_2=\gamma$, $\mu_3=\alpha$, $\nu_3=\delta$, and $\lambda=\alpha$;

$$a_{\alpha\gamma,\alpha\delta,\beta}+a_{\alpha\delta,\alpha\beta,\gamma}+a_{\alpha\beta,\alpha\gamma,\delta}-a_{\alpha\beta,\gamma\delta,\alpha}-a_{\alpha\gamma,\delta\beta,\alpha}-a_{\alpha\delta,\beta\gamma,\alpha}=0. \quad (8)$$

Using φ just defined we rewrite (8) as

$$\varphi^{\alpha}{}_{\delta\gamma\beta}+\varphi^{\alpha}{}_{\beta\delta\gamma}+\varphi^{\alpha}{}_{\gamma\beta\delta}=0. \quad (9)$$

Comparing Eqs. (7″) and (9), we conclude that all three terms in Eq. (9) are equal, and thus, $\varphi^{\alpha}{}_{\beta\gamma\delta}=0$ that means

$$a_{\alpha\beta,\gamma\alpha,\delta}+a_{\beta\gamma,\delta\alpha,\alpha}=0. \quad (10)$$

We now define $f_{\alpha\beta,\gamma}$ through the following equation (δ being known as soon as we have made a definite choice for $\alpha\beta$ and γ):

$$a_{\alpha\beta,\gamma\delta,\delta}=g_{\delta\delta}f_{\alpha\beta,\gamma}\quad\text{with}\quad f_{\alpha\beta,\gamma}=-f_{\beta\alpha,\gamma}. \quad (11)$$

With the help of (10) we find

$$a_{\alpha\beta,\alpha\gamma,\delta}=g_{\alpha\alpha}f_{\beta\gamma,\delta}. \quad (12)$$

Using (4) we also get

$$a_{\alpha\beta,\alpha\gamma,\alpha}=g_{\alpha\alpha}(f_{\beta\gamma,\alpha}+f_{\alpha\gamma,\beta}-f_{\alpha\beta,\gamma}). \quad (13)$$

So far we have only introduced $f_{\mu\nu,\rho}$ for $\mu\neq\nu\neq\rho$ with the property of being antisymmetric with respect to its first two indices. We were able to express in terms of these quantities our first three sets of unknowns. We turn now to the last set. For that purpose we write again (3) with $\mu_1=\alpha$, $\nu_1=\beta$, $\mu_2=\alpha$, $\nu_2=\gamma$, $\mu_3=\beta$, $\nu_3=\gamma$, and $\lambda=\alpha$. Using also (2) we get

$$a_{\alpha\gamma,\beta\gamma,\beta}=-a_{\alpha\beta,\gamma\beta,\gamma}. \quad (14)$$

We make a final use of (3) by setting $\mu_1=\alpha$, $\nu_1=\beta$, $\mu_2=\gamma$, $\nu_2=\delta$, $\mu_3=\alpha$, $\nu_3=\gamma$, and $\lambda=\beta$:

$$-g_{\beta\beta}a_{\gamma\delta,\alpha\gamma,\alpha}+g_{\gamma\gamma}a_{\alpha\beta,\delta\alpha,\beta}-g_{\alpha\alpha}a_{\gamma\delta,\gamma\beta,\beta}=0. \quad (15)$$

We want to combine (14) and (15). Let us call $\psi_{\alpha,\gamma\beta}=a_{\alpha\gamma,\beta\gamma,\beta}$. The last two equations read:

$$\psi_{\delta,\alpha\beta}=-\psi_{\delta,\beta\alpha}, \quad (14')$$

$$g_{\beta\beta}\psi_{\delta,\gamma\alpha}+g_{\gamma\gamma}\psi_{\delta,\alpha\beta}+g_{\alpha\alpha}\psi_{\delta,\beta\gamma}=0. \quad (15')$$

Because of the antisymmetry of $\psi_{\delta,\alpha\beta}$ in its last two indices, as soon as we choose δ we are left with only three quantities linked by the last equation. The most general solution is of the following form

$$\psi_{\delta,\alpha\beta}=g_{\alpha\alpha}h_{\delta,\beta}-g_{\beta\beta}h_{\delta,\alpha}$$

with arbitrary $h_{\delta,\beta}$. Indeed (14) and (15) are now identities. The three $h_{\delta,\alpha}, h_{\delta,\beta} h_{\delta,\gamma}$ only appear in the expression of the six $\psi_{\delta,\alpha\beta}=-\psi_{\delta,\beta\alpha}$, $\psi_{\delta,\alpha\gamma}=-\psi_{\delta,\gamma\alpha}$, $\psi_{\delta,\beta\gamma}=-\psi_{\delta,\gamma\beta}$, and the conditions for solving back for the h_δ, with one degree of arbitrariness, given the ψ_δ, $\cdots$ are precisely Eqs. (14) and (15).

At last we set $f_{\delta\alpha,\alpha}=-f_{\alpha\delta,\alpha}=h_{\delta,\alpha}$. These new f's are obviously independent of the ones previously defined. Together they build up $f_{\mu\nu,\rho}=-f_{\nu\mu,\rho}$, and we now have:

$$a_{\alpha\beta,\beta\gamma,\gamma}=\psi_{\alpha,\gamma\beta}=g_{\gamma\gamma}f_{\alpha\beta,\beta}-g_{\beta\beta}f_{\alpha\gamma,\gamma}. \quad (16)$$

Using (2) we can unite Eqs. (11), (12), (13), and (16) in a single expression:

$$\begin{aligned}a_{\mu_1\nu_1,\mu_2\nu_2,\lambda}&=f_{\mu_1\nu_1,\mu_2}g_{\nu_2\lambda}-f_{\mu_1\nu_1,\nu_2}g_{\mu_2\lambda}\\&-f_{\mu_2\nu_2,\mu_1}g_{\nu_1\lambda}+f_{\mu_2\nu_2,\nu_1}g_{\mu_1\lambda}-g_{\mu_1\nu_2}f_{\nu_1\mu_2,\lambda}\\&-g_{\nu_1\mu_2}f_{\mu_1\nu_2,\lambda}+g_{\mu_1\mu_2}f_{\nu_1\nu_2,\lambda}+g_{\nu_1\nu_2}f_{\mu_1\mu_2,\lambda}.\end{aligned} \quad (17)$$

Equations (2) and (3) are now identities. We are now in position to return to our Lie algebra. We set

$$L_{\mu\nu}=N_{\mu\nu}-f_{\mu\nu,\lambda}P^{\lambda} \quad (18)$$

with the same f's as those which appear in (17).

It is clear that $L_{\mu\nu}=-L_{\nu\mu}$ and that $L_{\mu\nu}$ and P^{λ} generate the same Lie algebra as the $N_{\mu\nu}$ and P^{λ} do. But in terms of these new operators, the commutation relations now read [using (17)]:

$$[P^{\lambda},P^{\rho}]=0,\ [P^{\lambda},L_{\mu\nu}]=\delta^{\lambda}_{\mu}P_{\nu}-\delta^{\lambda}_{\nu}P_{\mu},$$

$$[L_{\mu_1\nu_1},L_{\mu_2\nu_2}]=g_{\mu_1\nu_2}L_{\nu_1\mu_2}+g_{\nu_1\mu_2}L_{\mu_1\nu_2}-g_{\mu_1\mu_2}L_{\nu_1\nu_2}-g_{\nu_1\nu_2}L_{\mu_1\mu_2},$$

which are the usual commutation relations for the generators of the inhomogeneous Lorentz group. Hence, our lemma is proved.

Lemma 2: Let $\mathfrak{L}$ be a Lie algebra over R which as vector space is the direct sum of an ideal A and a subalgebra M; if A is a semisimple Lie algebra, then there exists an ideal N in $\mathfrak{L}$ isomorphic to M and such that $\mathcal{L}$ is the direct sum of the Lie algebras N and A,

$$\mathcal{L} = N \oplus A.$$

Proof of Lemma 2: Since A is an ideal in $\mathcal{L}$ if we choose a definite m in M, $[m, a]$ belongs to A and the correspondence $a \to [m, a]$ defined for every a in A is linear. Because of Jacobi identity it is even a derivation on A:

$$[b[m, a]] + [m[a, b]] + [a[b, m]] = 0$$

can be read

$$[m[a, b]] = [[m, a], b] + [a, [m, b]]. \qquad (19)$$

We now combine this fact, our hypothesis of the semisimplicity of A with the theorem quoted at the end of Sec. 2 to conclude that the derivation given by m is in fact an inner derivation of A; that means there exists an element $\varphi(m)$ belonging to A such that, for any $a \in A$,

$$[m, a] = [\varphi(m), a]. \qquad (20)$$

This element is unique for the equality $[\varphi_1(m), a] = [\varphi_2(m), a]$ which if valid for any $a \in A$ means that $\varphi_1(m) - \varphi_2(m)$ belongs to the center of A which for a semi-simple Lie algebra is reduced to zero. We now have a correspondence

$$\varphi : M \to A \quad m \to \varphi(m).$$

We prove that φ is an homomorphism. Because of the unicity of $\varphi(m)$, it is clear that φ is linear.

We use Jacobi identity and the fact that M is a subalgebra of L.

$$\begin{aligned} [[m_1, m_2], a] &= [\varphi([m_1 m_2]), a] \\ &= [m_1[m_2, a]] - [m_2[m_1, a]] = [\varphi(m_1)[\varphi(m_2), a]] \\ &\quad - [\varphi(m_2), [\varphi(m_1), a]] = [[\varphi(m_1), \varphi(m_2)], a]. \end{aligned}$$

The unicity of φ gives

$$\varphi([m_1, m_2]) = [\varphi(m_1), \varphi(m_2)]. \qquad (21)$$

This proves our statement that φ is indeed an homomorphism. Let us further note that

$$[m_1, \varphi(m_2)] = [\varphi(m_1), \varphi(m_2)]. \qquad (22)$$

We define now the subset N of $\mathcal{L}$. It is the image of the following application:

$$\psi: \ M \to \mathcal{L}, \qquad m \to \psi(m) = m - \varphi(m),$$

ψ is linear, and moreover, $\psi([m_1, m_2]) = [\psi(m_1), \psi(m_2)]$ which turns N into a subalgebra of $\mathcal{L}$ homomorphic to M. This last equality is a direct consequence of (21) and (22). It is also clear that $\psi(m) = 0$ implies $m = 0$. Indeed $\psi(m) = 0$ means $m = \varphi(m)$, but $m \in M$ and $\varphi(m) \in A$ means that $m = \varphi(m) = 0$ in virtue of our hypothesis that as a vector space, $\mathcal{L}$ is the direct sum of M and A. Hence, N is isomorphic to M. Further, any n in N is an image under ψ of an element in M: $n = \psi(m)$ so that $[n, a] = [m - \varphi(m), a] = 0$ because of (20).

It remains to show that any l belonging to $\mathcal{L}$ can be written in a unique way as $l = n + a$ with $n \in N$ and $a \in A$.

By hypothesis we know that $l = m + a' = [m - \varphi(m)] + [\varphi(m) + a'] = n + a$, and this n is obviously unique. This concludes the proof of Lemma 2.

4. PROOF OF THE THEOREM

We turn to the proof of the theorem. The assumptions made imply that the commutation relations between our operators have the following form:

$$\begin{aligned} &[P^\lambda, P^\mu] = 0, \qquad [P^\lambda, M_{\mu\nu}] = \delta^\lambda{}_\mu P_\nu - \delta^\lambda{}_\nu P_\mu, \\ &[M_{\mu_1\nu_1}, M_{\mu_2\nu_2}] = g_{\mu_1\nu_2} M_{\nu_1\mu_2} + g_{\nu_1\mu_2} M_{\mu_1\nu_2} \\ &\qquad\qquad - g_{\mu_1\mu_2} M_{\nu_1\nu_2} - g_{\nu_1\nu_2} M_{\mu_1\mu_2}, \\ &\qquad\qquad [A_i, P^\lambda] = 0, \qquad (23) \\ &[A_i, M_{\mu\nu}] = a_{i,\mu\nu}{}^{\rho\sigma} M_{\rho\sigma} + b_{i,\mu\nu}{}^l A_l + C_{i,\mu\nu,\lambda} P^\lambda, \\ &[A_i, A_j] = d_{ij}{}^{\rho\sigma} M_{\rho\sigma} + e_{ij}{}^l A_l + f_{ij,\lambda} P^\lambda, \end{aligned}$$

with obvious symmetry properties for the coefficients $a, \cdots, f$. First, let us prove that $a_{i,\mu\nu}{}^{\rho\sigma} = 0$ and $d_{ij}{}^{\rho\sigma} = 0$. We use the Jacobi identity

$$[P^\lambda, [A_i, A_j]] + [A_i, [A_j, P^\lambda]] + [A_j[P^\lambda, A_i]] = 0.$$

Because of (1) it reads

$$d_{ij}{}^{\rho\sigma}[P^\lambda, M_{\rho\sigma}] = 0;$$

hence, $d_{ij}^{\rho\sigma} = 0$. Again

$$[A_i, [M_{\mu\nu}, P^\lambda]] + [M_{\mu\nu}, [P^\lambda, A_i]] + [P^\lambda, [A_i, M_{\mu\nu}]] = 0;$$

$$a_{i,\mu\nu}^{\rho\sigma}[P^\lambda, M_{\rho\sigma}] = 0;$$

so also, $a_{i,\mu\nu}^{\rho\sigma} = 0$.

It is clear by inspection of (1) that the P^λ generate an ideal $\mathfrak{J}$ in $\mathcal{L}$, and so we can go to the factor Lie algebra $\mathcal{L}' = \mathcal{L}/\mathfrak{J}$. We denote by A_i' and $M_{\mu\nu}'$ the images of A_i and $M_{\mu\nu}$ under the mapping $\mathcal{L} \to \mathcal{L}/\mathfrak{J} = \mathcal{L}'$. Using the fact that the a's and d's are zero, we get for the commutation relations

$$[M'_{\mu_1\nu_1}, M'_{\mu_2\nu_2}] = g_{\mu_1\nu_2}M'_{\nu_1\mu_2} + g_{\nu_1\mu_2}M'_{\mu_1\nu_2} - g_{\mu_1\mu_2}M'_{\nu_1\nu_2} - g_{\nu_1\nu_2}M'_{\mu_1\mu_2},$$
$$[A'_i, M'_{\mu\nu}] = b^l_{i,\mu\nu}A'_l, \tag{24}$$
$$[A'_i, A'_j] = e^l_{ij}A'_l.$$

Equation (24) expresses the fact that the A'_i generate an ideal $\mathfrak{A}' \subset \mathfrak{L}$. We thus have proved the first two statements of the theorem.

Not only do the A'_i build up the ideal $\mathfrak{A}'$, but also the $M'_{\mu\nu}$ generate a subalgebra $\mathfrak{M}'$ and, as a vector space, $\mathfrak{L}'$ is the direct sum of $\mathfrak{M}'$ and $\mathfrak{A}'$. If we suppose that the algebra $\mathfrak{A}'$ is semisimple, we can make use of Lemma 2 which asserts that one can find real coefficients $k^i_{\mu\nu} = -k^i_{\nu\mu}$ such that defining:

$N'_{\mu\nu} = M'_{\mu\nu} - k^i_{\mu\nu}A'_i$, the $N'_{\mu\nu}$ also generate an ideal in $\mathfrak{L}'$, and the commutation relations in $\mathfrak{L}'$ are the same as in (24) with $M'_{\mu\nu}$ replaced by $N'_{\mu\nu}$ and the coefficients $b^l_{i,\mu\nu}$ put equal to zero.

This means, going back to the initial Lie algebra, that by replacing the $M_{\mu\nu}$ by $N_{\mu\nu} = M_{\mu\nu} - k^i_{\mu\nu}A_i$ we can manage to make the b coefficients disappear. Since $[A_i, P^\lambda] = 0$, $[N_{\mu\nu}, P^\lambda] = [M_{\mu\nu}, P^\lambda]$; on the other hand,

$$[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}] = [M_{\mu_1\nu_1}, M_{\mu_2\nu_2}] \text{ modulo } \mathfrak{I}.$$

This is the same statement as

$$[N'_{\mu_1\nu_1}, N'_{\mu_2\nu_2}] = [M'_{\mu_1\nu_1}, M'_{\mu_2\nu_2}] \quad \text{in} \quad \mathfrak{L}'.$$

In other words, the Lie algebra $\mathfrak{L}$ is also generated by P^λ, $N_{\mu\nu}$, A_i with the commutation relations now reading

$$[P^\lambda, P^\rho] = 0,$$
$$[P^\rho, N_{\mu\nu}] = \delta^\rho{}_\mu P_\nu - \delta^\rho_\nu P_\mu,$$
$$[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}] = g_{\mu_1\nu_2}N_{\nu_1\mu_2} + g_{\nu_1\mu_2}N_{\mu_1\nu_2} - g_{\mu_1\mu_2}N_{\nu_1\nu_2} - g_{\nu_1\nu_2}N_{\mu_1\mu_2} + m_{\mu_1\nu_1,\mu_2\nu_2,\rho}P^\rho, \tag{25}$$
$$[A_i, N_{\mu\nu}] = c_{i,\mu\nu,\lambda}P^\lambda,$$
$$[A_i, A_j] = e^l_{ij}A_l + f_{ij,\lambda}P^\lambda.$$

We now observe that P^λ and $N_{\mu\nu}$ generate a subalgebra in $\mathfrak{L}$, and the commutation relations are just the ones discussed in Lemma 1 where it was proved that there exist linear combinations

$$L_{\mu\nu} = N_{\mu\nu} - l_{\mu\nu,\lambda}P^\lambda = M_{\mu\nu} - k^i_{\mu\nu}A_i - l_{\mu\nu,\lambda}P^\lambda,$$

such that P^λ and $L_{\mu\nu}$ have the same commutation relations as the previous P^λ and $M_{\mu\nu}$ had. Again P^λ commutes with A_i so that

$[A_i, L_{\mu\nu}] = [A_i, N_{\mu\nu}]$, and also, $[P^\lambda, L_{\mu\nu}] = [P^\lambda, N_{\mu\nu}]$.

Hence, our algebra $\mathfrak{L}$ is still generated by P^λ, $L_{\mu\nu}$ and A_i, but we now have the commutation rules:

$$[P^\lambda, P^\rho] = 0,$$
$$[P^\rho, L_{\mu\nu}] = \delta^\rho_\mu P_\nu - \delta^\rho_\nu P_\mu,$$
$$[L_{\mu_1\nu_1}, L_{\mu_2\nu_2}] = g_{\mu_1\nu_2}L_{\nu_1\mu_2} + g_{\nu_1\mu_2}L_{\mu_1\nu_2} - g_{\mu_1\mu_2}L_{\nu_1\nu_2} - g_{\nu_1\nu_2}L_{\mu_1\mu_2}, \tag{26}$$
$$[A_i, L_{\mu\nu}] = c_{i,\mu\nu,\lambda}P^\lambda,$$
$$[A_i, A_j] = e^l_{ij}A_l + f_{ij,\lambda}P^\lambda.$$

We have not yet made full use of the Jacobi identities in $\mathfrak{L}$. Indeed we have to make sure that

$$[A_i, [L_{\mu\nu}, L_{\rho\sigma}]] + [L_{\mu\nu}, [L_{\rho\sigma}, A_i]] + [L_{\rho\sigma}, [A_i, L_{\mu\nu}]] = 0.$$

Using (26) we get

$$c_{i,\mu\nu,\rho}g_{\sigma\lambda} - c_{i,\mu\nu,\sigma}g_{\rho\lambda} - c_{i,\rho\sigma,\mu}g_{\nu\lambda} + c_{i,\rho\sigma,\nu}g_{\mu\lambda} = c_{i,\nu\rho,\lambda}g_{\mu\sigma} + c_{i,\mu\sigma,\lambda}g_{\nu\rho} - c_{i,\mu\rho,\lambda}g_{\nu\sigma} - c_{i,\nu\sigma,\lambda}g_{\mu\rho}. \tag{27}$$

We now try to find the most general solution of (27). Let us put in (27) $\mu = \rho \neq \nu \neq \sigma$ and $\lambda = \nu$. The result is

$$g_{\nu\nu}c_{i,\mu\sigma,\mu} = g_{\mu\mu}c_{i,\nu\sigma,\nu}.$$

Hence, $c_{i,\mu\sigma,\mu} = -g_{\mu\mu}p_{i,\sigma}$, where the $p_{i,\sigma}$ are some real numbers.

Again let us write (27) with $\mu = \rho \neq \nu \neq \sigma \neq \lambda$ which is possible since we have four values at our disposal. The result is $c_{i,\nu\sigma,\lambda} = 0$ if λ is different from ν and σ. Combining this result with the previous one, we get

$$c_{i,\mu\nu,\sigma} = g_{\nu\sigma}p_{i,\mu} - g_{\mu\sigma}p_{i,\nu}. \tag{28}$$

In writing (28) we have also made use of the fact that $c_{i,\mu\nu,\sigma} = -c_{i,\nu\mu,\sigma}$. Using (28) one can now check that (27) is an identity. We set $B_i = A_i - p_{i,\mu}P^\mu$. Now $[B_i, L_{\mu\nu}] = 0$, and since $[P^\lambda, A_i] = 0$ and $[P^\lambda, P^\rho] = 0$, we also have

$$[B_i, B_j] = [A_i, A_j] = e_{ij}{}^lA_l + f_{ij,\lambda}P^\lambda = e_{ij}{}^lB_l + f'_{ij,\lambda}P^\lambda,$$

with $f'_{ij,\lambda} = f_{ij,\lambda} + e_{ij}{}^lp_{l,\lambda}$. The last Jacobi identity, namely

$$[[B_iB_j], L_{\mu\nu}] + [[B_j, L_{\mu\nu}]B_i] + [[L_{\mu\nu}, B_i], B_j] = 0$$

tells us $f'_{ij,\lambda}[P^\lambda, L_{\mu\nu}] = 0$; that is, $f'_{ij,\lambda} = 0$.

Our proof is now complete since using the assumptions of the theorem we were able to show that starting with a set of P^λ, $M_{\mu\nu}$, and A_i satisfying (23) there exists a set of numbers $k_{\mu\nu}{}^i$, $l_{\mu\nu,\lambda}$, $p_{i,\mu}$ such that setting

$$L_{\mu\nu} = M_{\mu\nu} - k_{\mu\nu}{}^i A_i - l_{\mu\nu,\lambda} P^\lambda$$

and

$$B_i = A_i - p_{i,\mu} P^\mu, \tag{29}$$

the P^λ, $L_{\mu\nu}$, B_i generate the same Lie algebra as before, but their commutation relations now read

$$\begin{aligned} &[P^\lambda, P^\rho] = 0, \quad [P^\lambda, L_{\mu\nu}] = \delta^\lambda{}_\mu P_\nu - \delta^\lambda{}_\nu P_\mu, \\ &[L_{\mu_1\nu_1}, L_{\mu_2\nu_2}] = g_{\mu_1\nu_2} L_{\nu_1\mu_2} + g_{\nu_1\mu_2} L_{\mu_1\nu_2} \\ &\qquad\qquad - g_{\mu_1\mu_2} L_{\nu_1\nu_2} - g_{\nu_1\nu_2} L_{\mu_1\mu_2}, \\ &[B_i, P^\lambda] = [B_i, L_{\mu\nu}] = 0, \\ &[B_i, B_j] = e_{ij}{}^l B_l. \end{aligned} \tag{30}$$

Hence, we have succeeded in splitting our Lie algebra into a direct sum of two algebras, one generated by the P^λ and $L_{\mu\nu}$, call it $\mathcal{P}$, isomorphic to the one of the generators of the inhomogeneous Lorentz group, the second generated by the B_i, call it $\mathcal{A}$ with $\mathcal{A}$ semisimple. $\mathcal{A}$ is clearly isomorphic with $\mathcal{A}'$ introduced earlier. The theorem is proved.

5. REMARKS

The crucial point in Sec. 4 was in assuming that $\mathcal{A}'$ is semisimple. This of course has not much to do with any kind of physical assumption. However, as our discussion shows, if we keep the hypothesis that $[A_i, P^\lambda] = 0$, the only way to get a final answer different from the one given in the theorem is to find a Lie algebra $\mathcal{A}$ such that it should be impossible to interpret the derivations $A_i' \to [M_{\mu\nu}, A_i']$ as inner derivations. A typical case would be the following. Call $\mathfrak{D}$ the Lie algebra of derivations on $\mathcal{A}'$ and $\mathfrak{D}'$ the Lie algebra of inner derivations on $\mathcal{A}'$. If there exists a nontrivial homomorphism of the Lie algebra $\mathfrak{M}$ of the homogeneous Lorentz group in $\mathfrak{D}/\mathfrak{D}'$, then the last conclusion of the theorem would be false. By nontrivial homomorphism, we mean any homomorphism except the one which sends $\mathfrak{M}$ on $0 \in \mathfrak{D}/\mathfrak{D}'$. In our previous considerations we made the assumption of semisimplicity in order to assure $\mathfrak{D}/\mathfrak{D}' = 0$ which forced the homomorphism $\mathfrak{M} \to \mathfrak{D}/\mathfrak{D}' = 0$ to be the trivial one. We see that the theorem extends to the case where $\mathcal{A}$ has zero center and $\mathfrak{M} \to \mathfrak{D}/\mathfrak{D}'$ has to be the trivial homomorphism.

In conclusion one can point out the parallelism of this result with McGlinn's theorem which assumed that $[A_i, M_{\mu\nu}] = 0$. The outcome was then that the A_i generate a subalgebra $\mathcal{A}$ of $\mathcal{L}$; if this subalgebra was supposed to be semisimple (or more generally to have no abelian factor algebra[6]) then $[A_i, P^\lambda] = 0$ and $\mathcal{L}$ splits into $\mathcal{A} \oplus \mathcal{P}$.

ACKNOWLEDGMENTS

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[6] M. E. Mayer, H. T. Schnitzer, E. C. G. Sudarshan, R. Acharya, and M. Y. Han, Phys. Rev. **136**, B888 (1964).

The Gravitational Compass*

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A purely covariant approach to general relativity, using the equation of geodesic deviation, is adopted. The physical interpretation is essentially that due to Pirani, but instead of using clouds of particles to analyze the gravitational field, a "gravitational compass" is proposed which fulfills the same purpose. Particular attention is focussed on the different roles played by the matter and the free gravitational field. The latter splits up conveniently into a super-position of a transverse wave component, a longitudinal component, and a "Coulomb" field, all of which introduce "shearing" forces on the gravitational compass, while the matter contributes a general contraction. Applications to the Friedmann cosmological models and the problem of interacting gravitational waves are discussed.

1. INTRODUCTION

THE general principle of covariance makes the task of interpreting coordinate systems in general relativity an exceptionally difficult one. Indeed the only statement which general relativity makes about coordinates appears in the principle of equivalence, which is expressed mathematically by the fact that at any point of a Riemannian manifold the Christoffel symbols Γ^a_{bc} can be transformed away. A genuine gravitational field corresponds to a manifold in which this cannot be done everywhere simultaneously; that is, the covariant curvature tensor R_{abcd} must be nonvanishing. Consequently, we must concentrate our attention on this object if a satisfactory interpretation of general relativity is to be found.

The second fundamental assertion of general relativity is the postulation of the Einstein field equations,

$$G_{ab} = R_{ab} - 1/2Rg_{ab} = -T_{ab}. \qquad (1)$$

This equation reveals a peculiar dichotomy in the role of matter in general relativity; for on the left-hand side we have a geometrical object, to be measured by clocks and measuring rods, whereas on the right we have a dynamical quantity to be measured by pushing it around and observing the reaction. To resolve this difficulty we must realize that there are nongravitational fields such as electromagnetic and nuclear fields which give matter properties of solidity and hardness and allow us to carry out dynamical experiments. In a purely gravitational theory such as general relativity it is only the left-hand side of (0) that has a meaning; in this paper the point of view is adopted that the matter is given by the geometry rather than the other way around.

The full Riemann tensor has 20 independent components whereas R_{ab} has only 10. The following decomposition shows up the interdependence most clearly[1]:

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} + R_{a[c}g_{d]b} - \tfrac{1}{3}Rg_{a[c}g_{d]b}. \qquad (2)$$

The Weyl tensor C_{abcd} has the symmetries of a vacuum Riemann tensor; hence, it has 10 independent components which at any point are completely independent of the Ricci tensor components. We say, therefore, that it corresponds to the *free gravitational field*. Globally, however, the Weyl tensor and Ricci tensor are not independent, as they are connected by the differential Bianchi identities $R_{ab[cd;e]} = 0$. These identities determine the interaction between the free gravitational field and the field sources; they will be discussed in greater detail elsewhere.

In this paper the physical interpretation of the decomposition (2) is discussed in some detail. Pirani's approach[2] from the equation of geodesic deviation is adopted, but his "dust cloud" experiments are tidied up by using a tetrahedral arrangement of springs. It is to be emphasized, however, that the two descriptions are completely equivalent, the one adopted here only being preferred because of its simplicty. It is shown by means of the Petrov classification how the local free gravitational field is composed of the linear sum of three distinct components—a transverse wave, a longitudinal wave, and a Coulomb component, each having a characteristic effect on the gravitational compass. An application to the Friedmann cosmological models makes it clear that it is just in the existence of such *free*

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[1] Square brackets denote antisymmetrization, e.g., $A_{[abc]} = (1/3!)(A_{abc} - A_{acb} + A_{bca} - A_{bac} + A_{cab} - A_{cba})$.

[2] F. A. E. Pirani, Acta Phys. Polon. 15, 389 (1956).

gravitational fields (that is, in Weyl tensor components) that the Einstein theory differs most essentially from the Newtonian.

2. THE PHYSICAL INTERPRETATION OF THE PETROV CLASSIFICATION

Pirani and Schild[3] have given a physical interpretation of the Weyl tensor in terms of the deviation of null geodesics. While this approach has the advantage of being conform invariant, we shall consider only the deviation of timelike geodesics, as this appears to bring out the Petrov structure more clearly.

Consider a geodesic observer whose world line has unit tangent $u^a = dx^a/ds$

$$u_a u^a = -1, \qquad \dot{u}_a = u_{a;b}u^b = \left(\frac{d^2x^c}{ds^2} + \Gamma^c_{db}\frac{dx^d}{ds}\frac{dx^b}{ds}\right)g_{ca} = 0.$$

The signature is taken to be $+2$. Let δx^a be the displacement vector between neighboring geodesics $(u_a\delta x^a = 0)$. Then the equation of geodesic deviation[4] reads

$$\delta\ddot{x}^a = R^a_{bcd}u^bu^d\delta x^c. \tag{3}$$

Hence, the symmetric tensor $K_{ac} = R_{abcd}u^bu^d$ may be thought of as representing the gradient of the gravitational force in the instantaneous 3-space of the observer. Pirani[2] has suggested that the physical components of the Riemann tensor can be determined by experiments consisting of throwing up clouds of test particles and measuring the relative accelerations between them. A more straightforward and rather less cumbersome procedure is the following. Consider a tetrahedral arrangement of springs connecting three test particles to the central observer and each other, as shown in Fig. 1. At the instant of measurement the observer "drops" the apparatus (i.e., sends it on a geodesic tangent to his world line), and observes the strains on the springs. As the symmetric force distribution K_{ab} has six independent components, the strains on the six springs determine it completely as the solutions of an elementary problem in *statics*. When the strains on the diagonal springs S_{12}, S_{13}, S_{23} vanish, then the springs S_1, S_2, S_3 connecting the three test particles to the observer lie along the principal axes of K_{ab}. Thus the device maps out the local gravitational field and and an appropriate name for it is a "gravitational compass."

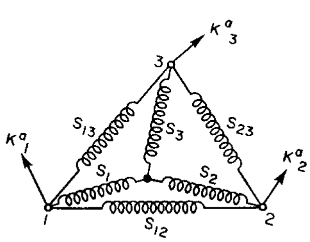

FIG. 1. The gravitational compass.

Substituting (2) into equation (3), we get

$$\delta\ddot{x}^a = C^a_{bcd}u^bu^d\delta x^c + \tfrac{1}{3}(R_{bd}u^bu^d)\delta x^a - \tfrac{1}{2}S^a_c\delta x^c, \tag{4}$$

where

$$h_{ab} = g_{ab} + u_au_b,$$
$$S_{ac} = h^b_ah^d_cR_{bd} - \tfrac{1}{3}h^{bd}R_{bd}h_{ac}.$$

Thus the force gradient K_{ab} is made up of a sum of three terms appearing on the right-hand side of (4). The first term is the contribution of the free gravitational field; it is to be thought of as introducing "shearing" forces, since it is symmetric and trace-free. The second term gives rise to an "expansive" force $\tfrac{1}{3}R_{bd}u^bu^dh^a_c$, while the last term represents a further shear introduced by the matter present.

In order to analyze the Weyl tensor contribution in (4), it is convenient to set up a tetrad of null vectors. To do this we supplement u^a with an orthonormal triad of spacelike vectors s^a, $\underset{1}{e^a}$, $\underset{2}{e^a}$, and put

$$k^a = u^a + s^a, \qquad t^a = (1/\sqrt{2})(\underset{1}{e^a} + i\underset{2}{e^a}),$$
$$m^a = \tfrac{1}{2}(s^a - u^a), \qquad \bar{t}^a = (1/\sqrt{2})(\underset{1}{e^a} - i\underset{2}{e^a}). \tag{5}$$

The null vectors k^a, m^a, t^a, $\bar{t}^a$ satisfy the quasiorthonormality conditions

$$k_am^a = t_a\bar{t}^a = 1, \qquad k_ak^a = k_at^a = m_am^a = m_at^a = t_at^a = 0, \tag{6}$$

and, following Sachs[5], we can decompose the (complex) Weyl tensor into null-tetrad components:

$$\begin{aligned} C_{abcd} + iC^*_{abcd} &= C^{(1)}V_{ab}V_{cd} \\ &+ C^{(2)}(V_{ab}M_{cd} + M_{ab}V_{cd}) \\ &+ C^{(3)}(M_{ab}M_{cd} + U_{ab}V_{cd} + V_{ab}U_{cb}) \\ &+ C^{(4)}(U_{ab}M_{cd} + M_{ab}U_{cd}) + C^{(5)}U_{ab}U_{cd}, \end{aligned} \tag{7}$$

where

$$V_{ab} = 2k_{[a}t_{b]}, \qquad U_{ab} = 2m_{[a}\bar{t}_{b]},$$
$$M_{ab} = 2k_{[a}m_{b]} + 2\bar{t}_{[a}t_{b]},$$

[3] F. A. E. Pirani and A. Schild, Bull. Acad. Polon. Sci. 9, 543 (1961).

[4] See for example, J. L. Synge and A. Schild, *Tensor Calculus* (Toronto University Press, Toronto, 1949).

[5] R. Sachs, Proc. Roy. Soc. (London) **A264**, 309 (1961).

and C^*_{abcd} is the dual of the Weyl tensor,

$$C^*_{abcd} = \tfrac{1}{2}(-g)^{\frac{1}{2}}\epsilon_{abij}C^{ij}{}_{cd},$$

where ϵ_{abcd} is the Levi-Civita alternating symbol.

The different Petrov types can be distinguished by the possibility of setting various combinations of the coefficients $C^{(\mu)}$ ($\mu = 1, \cdots, 5$) equal to zero in a suitable frame.[5] Thus Petrov type N is characterized by the fact that there exists a null vector k_a satisfying

$$C_{abcd}k^a = 0,$$

and by adapting the tetrad (5) to this null vector we have that $C^{(2)} = C^{(3)} = C^{(4)} = C^{(5)} = 0$. By a spacelike rotation $t^a \rightarrow e^{i\theta}t^a$ it is furthermore possible to make $C^{(1)}$ real. Any observer moving in a timelike direction u^a can set up a frame in which these conditions are satisfied, and the deviation of free test particles about him will be given by substituting into (3) and making use of (5):

$$\delta\ddot{x}^a = \tfrac{1}{2}C^{(1)}(\underset{1}{e}^a\underset{1}{e}_c - \underset{2}{e}^a\underset{2}{e}_c)\delta x^c. \tag{8}$$

This exhibits the well known transverse character of Petrov type N fields[6]; $\underset{1}{e}^a$ and $\underset{2}{e}^a$ are a pair of polarization axes lying in the plane orthogonal to s^a [Fig. 2(a)]. As this character is completely independent of the observer u^a, we call these fields *pure transverse gravitational waves*; s^a is the wave direction relative to the observer u^a.

In Petrov type-III fields there exists a null vector satisfying

$$k_a k_{[e}C^a{}_{b]cd} = 0,$$

and by choosing k_a in (5) along this direction we have $C^{(3)} = C^{(4)} = C^{(5)} = 0$. A spacelike rotation can again be used to make $C^{(2)}$ real. The second term on the right-hand side of (7) will then contribute a deviation

$$\delta\ddot{x}^a = \tfrac{1}{2}C^{(2)}(s^a\underset{1}{e}_c + \underset{1}{e}^a s_c)\delta x^c. \tag{9}$$

Thus the force distribution has a planar character similar to that in (8) but this time the plane contains the wave direction s_a; thus we may term this a *longitudinal wave component*. The polarization axes are tilted at 45° to $\underset{1}{e}^a$ and s^a [Fig. 2(b)]. In general there will be a transverse $C^{(1)}$ term such as (8) superimposed on the longitudinal component, but there is a special class of observers who can make $C^{(1)} = 0$. This is the class of observers for which u^a lies in the timelike 2-space spanned by the two Debever vectors k_a and m_a.

[6] F. A. E. Pirani, Phys. Rev. 105, 1089 (1957).

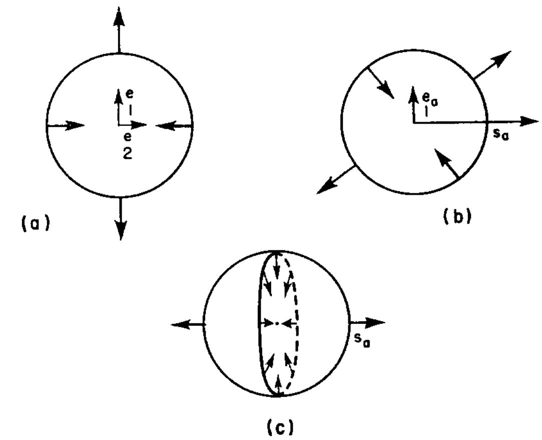

FIG. 2. The forces on the gravitational compass arising (a) from a transverse gravitational wave, (b) from the longitudinal component, and (c) from the Coulomb component.

The $C^{(3)}$ term of Eq. (7) gives rise to a contribution

$$\delta\ddot{x}^a = C_1^{(3)}[s^a s_c - \tfrac{1}{2}(\underset{1}{e}^a\underset{1}{e}_c + \underset{2}{e}^a\underset{2}{e}_c)]\delta x^c, \tag{10}$$

where

$$C^{(3)} = C_1^{(3)} + iC_2^{(3)}.$$

This force distribution has the effect of distorting a sphere of particles about the observer into an ellipsoid which has s^a as principal axis and is degenerate in the plane perpendicular to s^a [Fig. 2(c)]. This is typical of the behavior of particles falling in towards a central attracting body with the inverse square law (e.g., in the Schwarzschild solution, or in Newtonian theory). Hence, we call this term the *Coulomb part* of the field; the strength of the Coulomb field is given by the real part of $C^{(3)}$. In a field of Petrov type D it is always possible to find a frame in which $C^{(1)} = C^{(2)} = C^{(4)} = C^{(5)} = 0$, and hence, an observer for whom the force distribution will take on the appearance of Fig. 2(c).

In a Petrov type-II field the best we can do is to make $C^{(2)} = C^{(4)} = C^{(5)} = 0$, so that it can be viewed as a Coulomb field with an outgoing wave component superimposed. In the algebraically general case (Petrov type I) we must always get some contributions from $C^{(4)}$ and $C^{(5)}$ as well; at best it is possible to find a frame in which $C^{(2)} = C^{(4)} = 0$. To an observer in this frame the field will appear as a Coulomb field with incoming and outgoing transverse waves superimposed along the principal axis of the Coulomb field. This setting up of preferred frames with which to look at the gravitational field is completely analogous to the setting up of canonical frames in electrodynamics.[7]

3. THE EFFECT OF LORENTZ TRANSFORMATIONS

The preceding section demonstrates how an observer may use a gravitational compass to establish the force gradient of the gravitational field in his instantaneous 3-space. However, this is by no means the whole story. To find all physical components of the Weyl tensor we must consider observers with other velocities, that is to say, we must investigate the effects of a Lorentz transformation. Consider first a timelike rotation of the null tetrad in the (k, m) plane,

$$k'_a = \{(1+v)/(1-v)\}^{\frac{1}{2}} k_a,$$
$$m'_a = \{(1-v)/(1+v)\}^{\frac{1}{2}} m_a, \quad t'_a = t_a.$$

This corresponds to a simple Lorentz transformation to an observer u'^a traveling with velocity v in the s^a direction. Then

$$\begin{aligned} C'^{(1)} &= \tfrac{1}{2} C_{abcd} U'^{ab} U'^{cd} \\ &= C^{(1)}(1-v)/(1+v), \\ C'^{(2)} &= -\tfrac{1}{4} C_{abcd} U'^{ab} M'^{cd} \\ &= C^{(2)}\{(1-v)/(1+v)\}^{\frac{1}{2}}, \\ C'^{(3)} &= \tfrac{1}{8} C_{abcd} M'^{ab} M'^{cd} = C^{(3)}. \end{aligned} \tag{11}$$

Thus in Petrov type N, by increasing his speed in the wave direction s_a, the observer experiences a weakening of the wave amplitude by the factor $(1-v)/(1+v)$, and by a Lorentz transformation in the opposite direction he observes a corresponding increase in the amplitude. The same is true for the longitudinal type-III component except that the factor is $\{(1-v)/(1+v)\}^{\frac{1}{2}}$. $C^{(3)}$ is invariant; hence, the Coulomb strength is unaffected by such a transformation—the spring strains on the gravitational compass will be independent of the radial speed of the observer.

Consider now a transformation to a frame traveling with velocity v in the $\underset{1}{e}_a$ direction,

$$u'^a = (u^a + v\underset{1}{e}^a)\cdot(1-v^2)^{-\frac{1}{2}}.$$

In order to compensate for aberration it is necessary to supplement the simple Lorentz transformation with a rotation in the $(s_a, \underset{1}{e}_a)$ plane—that is, we want k^a to point in the s'^a direction. The desired Lorentz transformation can be achieved by the following rotation of the null tetrad:

$$\begin{aligned} k'^a &= k^a(1-v^2)^{\frac{1}{2}}, \\ t'^a &= t^a + k^a v/\sqrt{2}, \\ m'^a &= (1-v^2)^{-\frac{1}{2}} m^a - v\{2(1-v^2)\}^{-\frac{1}{2}}(t^a + \bar{t}^a) \\ &\quad - k^a\cdot v^2/2(1-v^2)^{\frac{1}{2}}. \end{aligned}$$

From Eqs. (6) and (7) we can find the transformation of the $C^{(\mu)}$'s under this Lorentz transformation. In particular if we are transforming from a canonical observer in a type-D field, we get

$$\begin{aligned} C'^{(1)} &= C^{(3)} 3v^2/(1-v^2), \\ C'^{(2)} &= C^{(3)} 3v/\{2(1-v^2)\}^{\frac{1}{2}}, \\ C'^{(3)} &= C^{(3)}. \end{aligned}$$

As $v \to 1$ the $C^{(1)}$ part dominates; for an observer rushing past a Schwarzschild source at a speed close to the fundamental velocity, the field will take on the appearance of a transverse gravitational wave.[8]

A vacuum gravitational field can be completely determined by means of these two Lorentz transformations. By using two gravitational compasses traveling at a relative velocity with respect to each other in the s_a direction, the observer can determine all components of the Weyl tensor except $C_2^{(3)}$. This is clear from (11) since $C^{(3)}$ is left invariant by this transformation and only the real part enters into the equation of geodesic deviation. A third compass traveling in the $\underset{1}{e}_a$ direction can be used to determine the imaginary part of $C^{(3)}$ as well. The presence of matter can in general be detected by a contraction on the gravitational compass since

$$K^a{}_a = R_{ab}u^a u^b = \tfrac{1}{2}(T_{11} + T_{22} + T_{33} - 3T_{00}),$$

which has a negative value for all physically realistic matter fields of nonzero rest mass. If matter is present, at least four compasses will be needed to determine the gravitational field, since the curvature tensor now has 20 independent components.

4. APPLICATIONS

A. Presence of a Null Electromagnetic Field

In this case

$$R_{ab} = -T_{ab} = -Ak_a k_b, \qquad k_a k^a = 0.$$

If u_a is any timelike vector with $k_a = u_a + s_a$, then

$$R_{ab}u^a u^b = -A, \qquad S_{ab} = -A(s_a s_b - \tfrac{1}{3}h_{ab}),$$

[7] J. L. Synge, *Relativity, the Special Theory* (North-Holland Publishing Company, Inc., Amsterdam, 1956).

[8] F. A. E. Pirani, Proc. Roy. Soc. (London) **A252**, 96 (1959).

and from (4) the total contribution to the geodesic deviation arising from the electromagnetic field is

$$-\tfrac{1}{2}A(\underset{1}{e}^a\underset{1}{e}_b + \underset{2}{e}^a\underset{2}{e}_b)\delta x^b.$$

That is, the electromagnetic field contributed an axially symmetric contraction about the wave direction. If the electromagnetic wave is accompanied by a pure transverse gravitational wave (cf. the $p\cdot p\cdot$ wave solutions of Kundt[9]), the directions of polarization of the gravitational wave will not be affected by the electromagnetic field, but the circle of accelerations will now develop an eccentricity.

B. Friedmann Cosmological Models

For a perfect fluid, the energy–stress tensor takes the form

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{12}$$

whence the field equations give

$$R_{ab}u^a u^b = -\tfrac{1}{2}(3p+\mu) + \Lambda, \qquad S_{ab} = 0,$$

where Λ is the cosmological constant. The Friedmann cosmological solutions are conformal to flat space and hence $C_{abcd} = 0$. In this case there is no free gravitational field present, and an observer traveling with the particles of the fluid will experience no shearing forces on his gravitational compass. He will observe contractive forces if $\Lambda < \frac{1}{2}(3p+\mu)$, expansive forces if $\Lambda > \frac{1}{2}(3p+\mu)$, and no forces if $\Lambda = \frac{1}{2}(3p+\mu)$ (Einstein's static model). The preferred frame determined by the distribution of matter in the Friedmann universe is the one for which the shearing forces S_{ab} vanish. Hence, by purely local experiments involving a gravitational compass an observer can determine the rest frame of the universe. In this sense, the Friedmann universe is "Machian." We see however that the expansion of the universe should not be looked upon as a gravitational effect at all, since it does not involve the Weyl tensor. The only force operative in the Friedmann models is the local contractive force

$$(-\tfrac{1}{2}(p+\tfrac{1}{3}\mu) + \tfrac{1}{3}\Lambda)\delta x^a,$$

which, in the case of vanishing pressure and cosmological constant, reduces to the Newtonian force due to small sphere of radius $(\delta x_a \delta x^a)^{\frac{1}{2}}$ and density $\rho = (3/8\pi)\gamma^{-1}\mu$ (γ = Newtonian gravitational constant). This accounts for the extraordinary sucess of the Newtonian theory in isotropic universes.[10] General relativity only differs radically from Newtonian theory when there is a free gravitational field (Weyl tensor) present; for example; there is nothing corresponding to the radiation fields in Newtonian theory. Hence the expansion of the universe is not due to any peculiar properties of gravitation in general relativity, but arises from one of the following possibilities:

(i) the initial conditions involve an initial expansion between neighboring particles (e.g., in the "big-bang" theory) or

(ii) the cosmological constant Λ is large (e.g., the deSitter universe or steady-state theory).

C. Interacting Gravitational Waves

Suppose we have two type-N gravitational waves traveling along distinct null directions k^a and m^a. The observer with the frame given by (5) sees these waves as traveling in opposite directions s_a and $-s_a$. If there were no interaction between the waves then the forces this observer obtains on his gravitational compass will be in the $(\underset{1}{e}_a, \underset{2}{e}_a)$ plane and the field strength should be the sum of the two wave amplitudes. Thus a solution representing two waves "passing through" each other would be expected to have, in this frame,

$$C_{abcd} = C^{(1)}V_{ab}V_{cd} + C^{(5)}U_{ab}U_{cd}.$$

Substituting into the vacuum Bianchi identities $C_{abcd;d} = 0$ and contracting with $k^a m^b U^{ce}$ gives

$$C^{(1)}k_{b;d}l^b k^d = C^{(1)}k_{b;d}l^b l^d = 0.$$

This is the condition that k_a is geodesic and shear-free (see Sachs[5]). There is a theorem of Goldberg and Sachs[11] which states the following:

"A vacuum metric is algebraically special if and only if it admits a shear-free null geodesic congruence."

Hence, in the above case the Weyl tensor must be algebraically special with k_a as a principal null direction. This contradicts $C^{(5)} \neq 0$. Hence, we cannot have a linear superposition of transverse gravitational waves if they are not running in the same direction, but the true nature of the interaction will remain obscure until exact solutions representing two waves in collision are found.

ACKNOWLEDGMENTS

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Algebraic Derivation of the Partition Function of a Two-Dimensional Ising Model*

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In the algebraic formulation of the Ising model, the partition function is expressed as the trace of a power V^M of the transfer operator V, or equivalently, as the sum of Mth powers of the eigenvalues of V. In the derivations of Kaufman, Onsager, and more recently of Schultz, Mattis, and Lieb (SML), the transfer operator is first reduced to a more amenable form for computation and then, in principle at least, diagonalized. For the infinite lattice, only the largest eigenvalue of V is needed, and this is all Onsager and SML compute. Kaufman finds all the eigenvalues and is thus able to write down the partition function for the finite lattice. In the present work we give an alternative derivation of the SML form for V, and show how the Kaufman result can be obtained from this form without actual diagonalization. Instead of diagonalizing V, the evaluation of the trace is done directly after assigning a simple representation to V.

I. INTRODUCTION

SINCE Onsager's celebrated solution of the two-dimensional Ising model in 1944,[1] there have been numerous alternative derivations of the Onsager result, a number of derivations of correlations and spontaneous magnetization, and some generalizations to other types of two dimensional lattices. Reviews of the work prior to 1960 are given in the articles of Domb[2] and Newell and Montroll.[3]

Roughly speaking, one can divide the methods used into two categories: the combinatorial approach, which essentially counts polygons on the lattice; and the algebraic approach. The combinatorial approach began in 1952 with the work of Kac and Ward.[4] Their approach has since been refined and been given a rigorous treatment,[5] and in its present form is the most popular method.[6] The popularity is undoubtedly due to the relative simplicity of the method compared with the original formidable algebraic approach of Onsager, and even with the simplified algebraic approach of Kaufman.[7] In recent times the algebraic approach has been given little attention. Recently however, Schultz, Mattis, and Lieb[8] (called SML hereinafter) have shown how the algebraic formulation of Onsager, leads to a many-fermion problem, the solution of which requires only an elementary knowledge of spin-$\frac{1}{2}$ and the second quantization formalism for Fermions. Much of the mysticism surrounding the algebraic method is clarified by SML; their steps are simple and their language more familiar. Of particular interest is their lucid discussion of correlations and spontaneous magnetization, and their new derivation of the transfer matrix formulation, which is the heart of the algebraic approach.

In brief outline, the problem is formulated algebraically as follows. Consider a square lattice consisting of M rows and N columns with a set of NM particles of two types arranged on the vertices of the lattice. If we let only nearest-neighbor particles interact, with interaction energy $-J_2(-J_1)$ between like particles in rows (columns) and $+J_2(+J_1)$ between unlike particles in rows (columns) and assign a coordinate $\mu_{m,n}$ to the vertex at the intersection of the nth row and mth column, with values $+1$ or -1 depending on the type of particle, the Hamiltonian for the system in configuration $\{\mu_{11}, \cdots, \mu_{MN}\}$,

$$H(\mu_{11}, \cdots, \mu_{MN}) = -J_1 \sum \mu_{m,n}\mu_{m+1,n} - J_2 \sum \mu_{m,n}\mu_{m,n+1}, \qquad (1)$$

defines the rectangular two-dimensional Ising model, and the problem is to calculate the partition function Z defined by

$$Z_{MN} = \sum_{\mu_{11}=\pm 1} \cdots \sum_{\mu_{MN}=\pm 1} e^{-\beta H(\mu_{11}, \cdots, \mu_{MN})}, \qquad (2)$$

where $\beta = (k_B T)^{-1}$, k_B being Boltzmann's constant, and T the absolute temperature.

For boundary conditions one usually takes $\mu_{M+m,n} = \mu_{m,n}$ and $\mu_{m,N+n} = \mu_{m,n}$, so that the lattice

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is assumed to be wrapped on a torus. In this case the sums in (1) are over n from 1 to N, and m from 1 to M. The toroidal boundary conditions will be used in the present work.

The reduction of Z_{MN} to the transfer operator form is well known, we merely quote the result. For detailed derivations, see for example the book by Huang[9] or SML.

The transfer operator V is defined by

$$V = \exp\left(K_2 \sum_{n=1}^{N} \tau_n^1 \tau_{n+1}^1\right) \exp\left(-K_1^* \sum_{n=1}^{N} \tau_n^2\right), \quad (3)$$

where

$$K_i = \beta J_i, \qquad i = 1, 2, \quad (4a)$$

K_1^* is related to K_1 through

$$\tanh K_1^* = e^{-2K_1}, \quad (4b)$$

and the operators τ_n^i, $i = 1, 2$, satisfy the relations

$$[\tau_n^i, \tau_{n'}^j]_- = \tau_n^i \tau_{n'}^j - \tau_{n'}^j \tau_n^i = 0, \quad (5a)$$

$$[\tau_n^i, \tau_n^j]_+ = \tau_n^i \tau_n^j + \tau_n^j \tau_n^i = 0, \quad (5b)$$

for $i \neq j = 1, 2,\ n, n' = 1, 2, \ldots, N$,

and

$$(\tau_n^i)^2 = I \quad \text{for} \quad i = 1, 2, \quad n = 1, 2, \cdots, N, \quad (5c)$$

with I the unit operator and 0 the null operator.

The first term in V arises from interactions in a row and the second from interactions between nearest-neighbor rows.

In terms of V, the partition function is given by

$$Z_{MN} = (2 \sinh 2K_1)^{\frac{1}{2}MN} \operatorname{Tr}(V^M). \quad (6)$$

A common feature of the Onsager, Kaufman, and SML derivations is that they all compute the eigenvalues of the transfer operator. Kaufman actually computes all the eigenvalues and is thus able to write down the partition function for the finite lattice from

$$Z_{MN} = (2 \sinh 2K_1)^{\frac{1}{2}MN} \sum_{n=1}^{2^N} \lambda_n^M. \quad (7)$$

Of more interest, phase transition-wise, is the infinite lattice where it is easily shown that the free-energy per particle is given by

$$-\beta^{-1} \lim_{\substack{M,N\to\infty \\ (M/N=\text{const})}} (MN)^{-1} \ln Z_{MN}$$
$$= -(2\beta)^{-1} \ln(2 \sinh 2K_1) - \beta^{-1} \lim_{N\to\infty} N^{-1} \ln \lambda_{\max} \quad (8)$$

with $\lambda_{\max}$ the largest eigenvalue of V.

[9] K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963), p. 349 ff.

For the infinite case it is therefore only necessary to find the largest eigenvalue, and this is done by Onsager and SML. Even so, one must in principle diagonalize V, and it is this part of the three calculations that is difficult, or at least complicated. In the present calculation, we do not find the eigenvalues of V. Instead we choose a representation of V in which the problem reduces to calculating traces of four-dimensional matrices of relatively simple structure. The calculation is given in the following section and the result obtained is Kaufman's expression for the finite problem. The passage to the infinite lattice can be followed in the manner suggested by Kaufman and is not included here. Our procedure, as a method for obtaining the infinite lattice result as well as the finite result, is, we feel, much less involved than the diagonalizing procedures of previous algebraic derivations. We remark also that our method can be used to evaluate the short- and long-range order (which are expressible as traces of operators). The calculations are straightforward and parallel closely those of SML so are not included here.

2. THE PARTITION FUNCTION

To express V in a more convenient form we define Fermi destruction and creation operators a_n and $a_n^\dagger (n = 1, 2, \cdots N)$, by

$$a_n + a_n^\dagger = \tau_1^2 \tau_2^2 \cdots \tau_{n-1}^2 \tau_n^3, \quad (9a)$$

$$a_n - a_n^\dagger = i \tau_1^2 \tau_2^2 \cdots \tau_{n-1}^2 \tau_n^1. \quad (9b)$$

In terms of these operators one can easily show that[10]

$$V^M = \tfrac{1}{2}(I + U) V_+^M + \tfrac{1}{2}(I - U) V_-^M, \quad (10)$$

where

$$U = \prod_{n=1}^{N} \tau_n^2 = \exp\left[i\pi \sum_{n=1}^{N} (a_n^\dagger a_n - 1)\right] \quad (11)$$

and

$$V_\pm = \prod_{n=1}^{N} \exp\left[K_2 (a_{n+1}^\dagger - a_{n+1})(a_n^\dagger + a_n)\right]$$
$$\times \prod_{n=1}^{N} \exp\left[-2K_1^* (a_n^\dagger a_n - 1)\right] \quad (12)$$

with the anticyclic boundary condition

$$a_{N+l} = -a_l \quad (13a)$$

holding for V_+, and the cyclic boundary condition

$$a_{N+l} = a_l \quad (13b)$$

[10] The steps follow closely those of Kaufman.

holding for V_-, and in deriving (10) we have made use of the fact that $P_\pm = \frac{1}{2}(I \pm U)$ are projection operators (which follows from $U^2 = I$), and act on orthogonal subspaces (i.e., $P_+P_- = 0$). Equation (12) was obtained by SML by a different method.

In the remainder of this section we will focus our attention on the V_+ term in (10). The steps in the calculation of the remaining three terms in (10) are similar to those given below and are summarized briefly at the end.

To simplify V_+ further we follow SML and transform to running wave operators through

$$a_n = N^{-\frac{1}{2}} e^{i\pi/4} \sum_q e^{iqn} \eta_q, \tag{14}$$

where the anticyclic condition (13a) requires that

$$q = \pm(2j-1)\pi/N \qquad j = 1, \cdots, N/2 \tag{15a}$$

[the cyclic condition (13b) requires that

$$q = 0, \pi, \pm 2j\pi/N, \quad j = 1, 2, \cdots, N/2 - 1] \tag{15b}$$

and for convenience we have chosen N to be even. Direct substitution of (14) into (12) gives

$$V_+ = \prod_{0<q<\pi} V_q, \tag{16}$$

where in terms of operators Σ_q^1, Σ_q^2, (and Σ_q^3 for completeness)

$$\begin{aligned} \Sigma_q^1 &= \eta_q^\dagger \eta_q + \eta_{-q}^\dagger \eta_{-q} - I, \\ \Sigma_q^2 &= \eta_{-q}\eta_q + \eta_q^\dagger \eta_{-q}^\dagger, \\ \Sigma_q^3 &= i(\eta_q^\dagger \eta_{-q}^\dagger - \eta_{-q}\eta_q), \end{aligned} \tag{17}$$

$$V_q = \exp\{2K_2[\cos q\Sigma_q^1 - \sin q\Sigma_q^2]\} \times \exp\{-2K_1^*\Sigma_q^1\}. \tag{18}$$

We will make use of the commutation relations for the Σ_q^i operators

$$\begin{aligned} &[\Sigma_q^i, \Sigma_{q'}^j]_- = -2i\delta_{qq'}\Sigma_q^k, \ (ijk) \quad \text{cyclic } (123) \\ &\text{and} \qquad [\Sigma_q^i, \Sigma_q^j]_+ = 0 \quad \text{for} \quad i \neq j. \end{aligned} \tag{19}$$

These follow directly from (17) and the Fermi anticommutation relations for the η_q -operators. We remark in passing that the sequence of equations (16)–(18)–(19) was obtained originally by Onsager using a slightly more involved method.

In all previous algebraic derivations, $V_\pm$ were diagonalized (simultaneously because of their commutability). Let us show that for the purpose of computing the trace, the most convenient representation is not the diagonal one but one based on the occupation number representation of the η_q-Fermi operators. Thus, since the eigenvalues of $\eta_{\pm q}^\dagger \eta_{\pm q}$ are 0 and 1 (with equal degeneracy), the eigenvalues of Σ_q^1 are 1, -1, 0, 0 (equal degeneracy). A representation of the Σ-operators in which Σ_q^1 is diagonal is therefore [from (19)]

$$\Sigma_q^i = I_4 \otimes \cdots \otimes I_4 \otimes \sigma^i r^+ \otimes I_4 \otimes \cdots \otimes I_4, \qquad i = 1, 2, 3, \tag{20}$$

where

$$\sigma^i = \begin{bmatrix} \tau^i & 0_2 \\ 0_2 & \tau^i \end{bmatrix}, \qquad r^+ = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix}. \tag{21}$$

τ^i are the conventional Pauli matrices

$$\tau^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau^3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{22}$$

I_4 and I_2 are the four- and two- dimensional unit matrices, respectively, 0_2 is the two-dimensional null matrix, and in the direct product (denoted by $\otimes$), there are $N/2$ terms, with the $\sigma^i r^+$ occuring in the jth ($q = 2j - 1$) place.

Using the property

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{23}$$

of direct product, we then have

$$V_+ = \bigotimes_{j=1}^{N/2} V_{2j-1} = \bigotimes_{p+} V_{p+}, \tag{24}$$

where

$$V_{p+} = \exp\{2K_2[\cos(p+)\sigma_+^1 - \sin(p+)\sigma_+^2]\} \times \exp\{-2K_1^*\sigma_+^1\} \tag{25}$$

and

$$\sigma_+^i = \sigma^i r^+. \tag{26}$$

If we then use the property

$$\mathrm{Tr}\,(A \otimes B) = \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B), \tag{27}$$

we have

$$\mathrm{Tr}\,(V_+^M) = \prod_{j=1}^{N/2} \mathrm{Tr}\,(V_{2j-1}^M). \tag{28}$$

The problem has now been reduced to the evaluation of traces of four-dimensional matrices with relatively simple structure. Let us consider V_p (p will henceforth be used for $p+$). Since the σ^i matrices satisfy the commutation relations

$$[\sigma_+^i, \sigma_+^j] = -2i\sigma_+^k \quad (ijk) \text{ cyclic } (123). \tag{29}$$

V_p can be written in the form

$$V_p = \exp\left(\sum_{i=1}^3 c_p^i \sigma_+^i\right). \tag{30}$$

If we then use the anticommutation relations

$$[\sigma_+^i, \sigma_+^j]_+ = 2\delta_{ij} r^+ \tag{31}$$

expansion of the exponential (30) gives

$$V_p = r^- + r^+ \cosh \gamma_p + \left(\sum_{i=1}^{3} c_p^i \sigma_+^i\right) \frac{\sinh \gamma_p}{\gamma_p}, \tag{32}$$

where

$$r^- = I_4 - r^+ \tag{33}$$

and γ_p is defined by

$$\gamma_p^2 = \sum_{i=1}^{3} (c_p^i)^2. \tag{34}$$

If we also expand the exponentials in (18) in a similar manner, we obtain

$$\begin{aligned} V_p &= r^- \\ &+ r^+[\cosh 2K_2 \cosh 2K_1^* - \sinh 2K_2 \sinh 2K_1^* \cos p] \\ &+ \sigma_+^1[\sinh 2K_2 \cosh 2K_1^* \cos p - \sinh 2K_1^* \cosh 2K_2] \\ &+ \sigma_+^2[-\sinh 2K_2 \cosh 2K_1^* \sin p] \\ &+ \sigma_+^3[\sinh 2K_1^* \sinh 2K_2 \sin p]. \end{aligned} \tag{35}$$

This expression must be identical with (32), so equating coefficients of r^+ gives the connection

$$\cosh \gamma_p = \cosh 2K_2 \cosh 2K_1^* - \sinh 2K_2 \sinh 2K_1^* \cos p \tag{36}$$

between the c_p^i parameters in (30) and K_2, K_1^*, and p. It turns out that (36) is the only relation needed to evaluate the trace (28). The steps needed are as follows. From (30) we have

$$\begin{aligned} (V_p)^M &= \exp\left(\sum_{i=1}^{3} (Mc_p^i)\sigma_+^i\right) \\ &= r^- + r^+ \cosh (M\gamma_p) + \left(\sum_{i=1}^{3} c_p^i \sigma_+^i\right) \frac{\sinh (M\gamma_p)}{\gamma_p} \end{aligned} \tag{37}$$

and recalling expressions (21) and (33) for r^+ and r^-, and noting that $\mathrm{Tr}\,(\sigma_+^i) = 0$, we have

$$\mathrm{Tr}\,(V_p^M) = 2[1 + \cosh (M\gamma_p)] = 4 \cosh^2 (\tfrac{1}{2}M\gamma_p). \tag{38}$$

To evaluate $\mathrm{Tr}\,(UV_+^M)$ in (10) we use the representation (20) to get

$$U_+ = \bigotimes_{p+} (-e^{i\pi\sigma_+'}) = \bigotimes_{p+} (r^+ - r^-) \tag{39}$$

and then from (23) and (27)

$$\mathrm{Tr}\,(UV_+^M) = \prod_{p+} \mathrm{Tr}\,[(r^+ - r^-)V_{p+}^M] \tag{40}$$

where, using $r^+r^- = 0_4 = r^-\sigma_+^i$, $(r^+)^2 = r^+$, and (37),

$$\begin{aligned} \mathrm{Tr}\,[(r^+ - r^-)V_{p+}^M] &= 2[-1 + \cosh (M\gamma_p)] \\ &= 4 \sinh^2 (\tfrac{1}{2}M\gamma_p). \end{aligned} \tag{41}$$

To evaluate the corresponding minus quantities $\mathrm{Tr}\,(V_-^M)$ and $\mathrm{Tr}\,(UV_-^M)$ in (10), one uses the representation (20) with $\sigma^i r^-$ in place of $\sigma^i r^+$ [to preserve the orthogonality of the two terms in (10)]. The above calculation then goes through with essentially just a minus sign in place of a plus [although some special care must be taken with the $q = 0$ and $q = \pi$ terms (15b) arising from the cyclic boundary conditions (13b).]

If we now define γ_k to be the positive solution of Eq. (36), that is, the positive solution of ($p = k\pi/N$)

$$\cosh \gamma_k = \cosh 2K_2 \cosh 2K_1^* - \sinh 2K_2 \times \sinh 2K_1^* \cos\left(\frac{k\pi}{N}\right) \qquad k = 0, 1, \cdots \tag{42}$$

and note that $\gamma_{2N-k} = \gamma_k$ for $k = 0, 1, \cdots, N$, we have finally, after combining (10), (28), (38), (40), and (41) and the corresponding minus results, the Kaufman expression for the partition function of the finite lattice,

$$\begin{aligned} Z_{MN} &= (2 \sinh 2K_1)^{\frac{1}{2}MN}\, \mathrm{Tr}\,(V^M) \\ &= \tfrac{1}{2}(2 \sinh 2K_1)^{\frac{1}{2}MN} \\ &\quad \times \left\{\prod_{j=1}^{N} 2 \cosh\left(\frac{M\gamma_{2j-1}}{2}\right) + \prod_{j=1}^{N} 2 \sinh\left(\frac{M\gamma_{2j-1}}{2}\right)\right. \\ &\quad \left. + \prod_{j=1}^{N} 2 \cosh\left(\frac{M\gamma_{2j}}{2}\right) + \prod_{j=1}^{N} 2 \sinh\left(\frac{M\gamma_{2j}}{2}\right)\right\}. \end{aligned} \tag{43}$$

The partition function for the infinite lattice can be obtained straightforwardly from (43) in the manner suggested by Kaufman. The reader is referred to Kaufman's article for details.

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Momentum-Transfer Theorem for Inelastic Processes

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Recently it has been shown that for potential scattering, the well-known optical theorem—relating the total cross section to the imaginary part of the forward scattering amplitude—can be generalized to yield a "momentum-transfer cross-section theorem." The present paper further generalizes the previous potential scattering result. Specifically, it appears that the momentum-transfer cross-section theorem is valid also for many-particle systems, wherein inelastic processes occur. Although this last assertion probably holds quite generally, a proof is given only for the collisions of electrons with atomic hydrogen. The proof takes into account electron indistinguishability, as well as the possibility that the incident electron ionizes the atom, but assumes the forces are not spin-dependent.

I. INTRODUCTION AND SUMMARY

RECENTLY[1] I have shown that for potential scattering, the momentum-transfer cross section

$$\sigma_d = \int d\mathbf{n}'(1 - \mathbf{n}\cdot\mathbf{n}')\ |A(\mathbf{n} \rightarrow \mathbf{n}')|^2 \tag{1}$$

can be expressed in the form

$$\sigma_d = \frac{1}{2E}\int d\mathbf{r}\ \Psi^* \frac{\partial V}{\partial z}\Psi. \tag{2}$$

In the above equations: $A(\mathbf{n} \rightarrow \mathbf{n}')$ is the amplitude for elastic scattering from initial direction $\mathbf{n}$ to final direction $\mathbf{n}'$; $E = \hbar^2k^2/2m$ is the kinetic energy; the Hamiltonian is

$$H = T + V = (-\hbar^2/2m)\nabla^2 + V(\mathbf{r}); \tag{3}$$

the potential $V(\mathbf{r})$ is not necessarily spherically symmetric, i.e., $V(\mathbf{r})$ need not equal $V(r)$; the wavefunction Ψ satisfies

$$(H - E)\Psi = 0 \tag{4}$$

subject to the boundary condition (when $\mathbf{n}$ is along the z direction)

$$\Psi = e^{ikz} + \Phi(\mathbf{r}), \tag{5}$$

where

$$\lim_{r\rightarrow\infty\|\mathbf{n}'} \Phi = A(\mathbf{n} \rightarrow \mathbf{n}')e^{ikr}r^{-1}; \tag{6}$$

when $\mathbf{n}$ is not parallel to z, $\partial V/\partial z$ in Eq. (2) is replaced by $\mathbf{n}\cdot\text{grad}\ V$.

For potential scattering the result (2) in many respects is a generalization of the optical theorem

$$\sigma = \int d\mathbf{n}'\ |A(\mathbf{n} \rightarrow \mathbf{n}')|^2 = \frac{4\pi}{k}\ \text{Im}\ A(\mathbf{n} \rightarrow \mathbf{n}). \tag{7}$$

It is known, however, that the optical theorem remains valid in many-particle collisions involving inelastic processes. Similarly, it appears that the momentum-transfer cross-section theorem (2) remains valid even when inelastic processes can occur. Of course, some modification of the right side of (2) is necessary in a many-particle collision. Also, one must generalize the definition (1) of the momentum-transfer cross section, σ_d.

Because a proof of the momentum-transfer cross-section theorem for arbitrarily complicated colliding systems would be awkward and hard to follow (mainly because the notation gets correspondingly complicated), I shall content myself here with carrying out the proof for the simple case of e–H scattering. In this case the momentum-transfer theorem has the form

$$\sigma_d = \frac{1}{2E_0}\int d\mathbf{r}\ \Psi^* \frac{\partial V}{\partial z_1}\Psi, \tag{8}$$

where $E_0 = \hbar^2k_0^2/2m$ is the incident kinetic energy; z_1 is the z coordinate of one of the two electrons in the system; and the quantities V, Ψ, σ_d are defined, respectively, by Eqs. (12), (14), and (56) below. This proof for e–H scattering makes it fairly obvious that a similar momentum-transfer theorem holds for electron scattering by more complicated atoms, and makes it plausible that a corresponding momentum-transfer theorem continues to hold for collisions between more complex aggregates of fundamental particles, e.g., for molecule–molecule scattering.

In connection with the above paragraph, the following remarks, concerning assumptions made in the proof, should be noted. The proof includes the effects of particle indistinguishability and electron exchange, i.e., the wavefunction is antisymmetric under exchange of electron space and spin coordinates. However, the spin-dependent part of the wavefunction is factored out, i.e., it is assumed that

[1] E. Gerjuoy, J. Math. Phys. 6, 993 (1965).

all components of the total spin are separately conserved, which in turn implies that the Hamiltonian is spin independent. There is little doubt that a momentum-transfer theorem remains valid for spin-dependent interactions, but carrying through the proof would require taking into account the properties of the eigenfunctions under time reversal; considering only the spatially dependent part of the wavefunction, as is done here, avoids this complication. Another complication which is ignored in the following proof of (8) is the effects of Coulomb forces on the asymptotic behavior of the continuum wavefunction solving the many-particle Schrödinger equation. More precisely, although ionization is included in the possible inelastic processes contributing to momentum transfer, it is assumed that the Hamiltonian is effectively a free-particle Hamiltonian when the particles are infinitely separated. It is easily seen that this assumption is inconsequential for (8) when the free-particle plane waves can be replaced by Coulomb functions as, e.g., in excitation of H^- by electrons, or ionization of H^- by a neutral particle. In more complex situations, e.g., ionization of H^- or H by electrons, there is no reason to think the momentum-transfer theorem fails, but it must be admitted that the detailed asymptotic behavior of the wavefunction has not been examined in circumstances such as these, where two or more charged particles go out to infinity in the center of mass system. Finally, the proof wholly ignores radiative processes.

The possible utility of (8) has been discussed previously.[1] Bearing on its utility, and relevant also to the discussion of the preceding paragraph, is the fact that the right side of (8) apparently diverges whenever electrons are incident on ions, e.g., H^-. The source of the divergence can be understood by examining elastic scattering in a fixed Coulomb potential $V = C/r$. Substituting (5) in (2), which now is applicable, one sees that integration over angles annihilates the matrix element of $\partial V/\partial z = -C\cos\theta/r^2$ between e^{ikz} and e^{-ikz}. The matrix element of $\partial V/\partial z$ between e^{-ikz} and Φ need not vanish, however, and in this matrix element the integral over r is divergent at $r = \infty$. Moreover, this divergence is to be expected, because for Coulomb scattering, directly from the fundamental definition (1),

$$\sigma_d \approx \int_0^\pi d\theta \sin\theta(1 - \cos\theta)\csc^4 \tfrac{1}{2}\theta \qquad (9)$$

diverges logarithmically at $\theta = 0$.

[2] J. O. Hirschfelder, J. Chem. Phys. 33, 1462 (1960).

The following remarks are also worth noting. The proof of (8) given here indicates that in a sense the momentum-transfer cross-section theorem is a generalization—to continuum eigenfunctions—of the so-called hypervirial theorems.[2] In fact the proof of (8) is based on a wholly time-independent (wherein transition probabilities are never explicitly introduced) treatment[3] of many-particle collisions involving rearrangement. In this treatment the cross section is computed, using Green's theorem, from the flow of probability current across the surface at infinity in the $3n$-dimensional space spanned by $\mathbf{r}_1, \cdots, \mathbf{r}_n$, where the collision involves n particles in all, and $\mathbf{r}_i$ is the position vector of the ith particle. The time-independent treatment has the advantage that it makes explicit many of the mathematical assumptions (e.g., those mentioned in the two preceding paragraphs) employed in proving Eq. (8).

II. REVIEW OF TIME-INDEPENDENT FORMALISM

Especially when ionization can occur, to make the proof of the momentum-transfer theorem understandable, it is desirable to review some results of the time-independent treatment. As explained above, I confine my attention to the scattering of electrons by atomic hydrogen in the ground $1s$ state ϕ_0. The atomic hydrogen eigenfunction $\phi_i(\mathbf{r})$, of energy ϵ_i, obeys

$$[(-\hbar^2/2m)\nabla^2 - e^2/r]\varphi_i = \epsilon_i\varphi_i. \qquad (10)$$

The spatially dependent part of the total wavefunction describing the collision is $\Psi(\mathbf{r}_1, \mathbf{r}_2)$, which satisfies Eq. (4) with

$$H = \frac{-\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2), \qquad (11)$$

$$V = \frac{-e^2}{r_1} - \frac{e^2}{r_2} + \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \qquad (12)$$

and obeys

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\Psi(\mathbf{r}_2, \mathbf{r}_1). \qquad (13)$$

The upper sign in Eq. (13) applies to singlet scattering, the lower sign to triplet scattering. In what follows, I shall use the plus sign only, but it is easily verified that the proof can be just as readily carried through for triplet scattering.

Outgoing Current and the Total Cross Section

Ignoring the long-range character of the potential, the singlet Ψ can be written in the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(\mathbf{r}_2) + e^{ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0(\mathbf{r}_1) + \Phi(\mathbf{r}_1, \mathbf{r}_2), \qquad (14)$$

[3] E. Gerjuoy, Ann. Phys. 5, 58 (1958).

where $\Phi(\mathbf{r}_1, \mathbf{r}_2)$ is everywhere outgoing and obeys

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = \Phi(\mathbf{r}_2, \mathbf{r}_1). \tag{15}$$

The everywhere-outgoing property implies

$$\lim_{r_1\to\infty \| \mathbf{n}'} \int d\mathbf{r}_2 \phi_j^*(\mathbf{r}_2)\Phi(\mathbf{r}_1, \mathbf{r}_2) = A_j(\mathbf{n}\to\mathbf{n}') \frac{e^{ik_jr_1}}{r_1}, \tag{16}$$

where

$$E = \hbar^2 k_0^2/2m + \epsilon_0 = \hbar^2 k_j^2/2m + \epsilon_j. \tag{17}$$

It is easily seen that, as one expects for singlet scattering,

$$A_j = f_j + g_j, \tag{18}$$

where f_j and g_j are, respectively, the ordinary and exchange amplitudes for collisions leaving the atom in the state ϕ_j.

Equation (16) yields no information about the behavior of Φ when r_1, r_2 each become infinite. However, the everywhere-outgoing property also implies[3]

$$\lim_{\substack{r_1\to\infty\|\mathbf{n}_1'\\ r_2\to\infty\|\mathbf{n}_2'\\ r_1/r_2=q=\text{constant}}} \Phi(\mathbf{r}_1, \mathbf{r}_2) = A(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2') \frac{e^{iKr}}{r^{5/2}}, \tag{19}$$

where

$$E = \frac{\hbar^2K^2}{2m} = \frac{\hbar^2k_1'^2}{2m} + \frac{\hbar^2k_2'^2}{2m}, \tag{20}$$

$$r = (r_1^2 + r_2^2)^{\frac{1}{2}} = r_1(1+q^2)^{\frac{1}{2}}, \tag{21}$$

$$\begin{aligned} \mathbf{k}_1' &= Kr_1\mathbf{n}_1'/r = K(1+q^2)^{-\frac{1}{2}}\mathbf{n}_1', \\ \mathbf{k}_2' &= Kr_2\mathbf{n}_2'/r = Kq(1+q^2)^{-\frac{1}{2}}\mathbf{n}_2'. \end{aligned} \tag{22}$$

The total cross section, including ionization as well as excitation, is[3]

$$\sigma = \frac{m}{2\hbar k_0}\int_\infty dS\boldsymbol{\nu}\cdot\mathbf{J} \tag{23}$$

integrated over the surface of the six dimensional sphere at infinity in $\mathbf{r}_1, \mathbf{r}_2$ space, where the six-dimensional current vector $\mathbf{J}$ has components

$$\begin{aligned} \mathbf{J}_1 &= (\hbar/2mi)(\Phi^*\nabla_1\Phi - \Phi\nabla_1\Phi^*), \\ \mathbf{J}_2 &= (\hbar/2mi)(\Phi^*\nabla_2\Phi - \Phi\nabla_2\Phi^*). \end{aligned} \tag{24}$$

In (24) $\mathbf{J}_1$ represents the three components of $\mathbf{J}$ along $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$, i.e., along the usual right-handed basis defining the $\mathbf{r}_1$ subspace of $\mathbf{r}_1, \mathbf{r}_2$ space. Similarly, $\mathbf{J}_2$ represents the three components of $\mathbf{J}$ along $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$. Correspondingly, in (23) the outward drawn six-dimensional normal to the sphere at infinity has components

$$\begin{aligned} \boldsymbol{\nu}_1 &= \mathbf{r}_1/r = (r_1/r)\mathbf{n}_1', \\ \boldsymbol{\nu}_2 &= \mathbf{r}_2/r = (r_2/r)\mathbf{n}_2'. \end{aligned} \tag{25}$$

Excitation and Ionization Cross Sections

The result (23) is basic to the time-independent treatment of many-particle collisions and is not evident. In fact, Eq. (23) amounts to accepting the postulate that (24) represents the current operator conserving probability flux in many-particle collisions, just as the usual formula (7) for the total cross section in potential scattering implies acceptance of the usual one-particle current operator [of which Eq. (24) is the obvious generalization]. Nevertheless, the correctness of (23) is not in question since it can be shown that the accepted expressions for the rates of excitation and ionization follow from (23).

To amplify this last assertion, note that on the sphere at infinity the surface elements dS (and corresponding $\boldsymbol{\nu}$) are of two essentially different types, namely: surface elements dS where one of r_1, r_2 is infinite, but not the other; and those dS forming a manifold of higher dimensionality than the first type, where r_1 and r_2 are each infinite. It has been proved[3] that the contribution to (23) from surface elements of the first type, with $r_1 \to \infty$ and r_2 finite, reduces to

$$\frac{m}{2\hbar k_0}\sum_j \int_\infty dS_1\, \mathbf{n}'\cdot\mathbf{J}_j' \tag{26}$$

integrated over the surface S_1 of the three-dimensional sphere at infinity in $\mathbf{r}_1$-space, where

$$\mathbf{J}_j' = (\hbar/2mi)(Z_j^*\nabla_1 Z_j - Z_j\nabla_1 Z_j^*) \tag{27}$$

and

$$Z_j(\mathbf{r}_1) = \int d\mathbf{r}_2\, \phi_j^*(\mathbf{r}_2)\Phi(\mathbf{r}_1, \mathbf{r}_2) \tag{28}$$

integrated over all $\mathbf{r}_2$. Of course in (26) $\mathbf{n}'$ is the normal to dS_1, and $\mathbf{J}_j'(\mathbf{r}_1)$ is evaluated at infinite $\mathbf{r}_1 = r_1\mathbf{n}'$. Using (16), therefore, (26) yields

$$\sum_j \frac{k_j}{2k_0}\int d\mathbf{n}'\, |A_j(\mathbf{n}\to\mathbf{n}')|^2 = \tfrac{1}{2}\sigma_{\text{ex}}, \tag{29}$$

where σ_{ex} obviously is the total cross section for excitation, including elastic scattering ($j = 0$). The left side of (29) is only half σ_{ex} because (26) has not included the contribution to (23) from surface elements with $r_2 \to \infty$ and r_1 finite; by virtue of (15) the contributions from $r_1 \to \infty$, r_2 finite and

$r_2 \to \infty$, r_1 finite must be equal. Correspondingly, one sees that the right side of (23), which must represent the total outgoing current divided by the incident current per unit area is correctly multiplied by $(2\hbar k_0/m)^{-1}$, because each of the first two terms on the right side of (14) corresponds to an incident current density $\hbar k_0/m$.

The contribution to (23) from surface elements dS where r_1 and r_2 are each infinite must be the ionization cross section σ_{ion}. In fact, for K real [$E > 0$ in (17) and therefore capable of ionizing the atom], this contribution is, using Eqs. (19) and (21)–(25),

$$\sigma_{\text{ion}} = \frac{1}{2k_0}\int \frac{dS}{r^5} K\, |A(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2')|^2 \tag{30a}$$

$$= \frac{1}{2k_0}\int dk_2'\, d\mathbf{n}_1'\, d\mathbf{n}_2' \frac{k_2'^2 k_1'}{K^3}\, |A(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2')|^2, \tag{30b}$$

where I have used[3]

$$dS = [r^5 q^2/(1+q^2)^3]\, dq\, d\mathbf{n}_1'\, d\mathbf{n}_2'. \tag{31}$$

The right side of Eq. (30b), which still is subject to Eq. (20), is not altered in value if one multiplies by $\delta(E' - E)$ and then integrates over an infinitesimal range dE' about $E' = E$. Thus, using Eq. (20) to find dE' in terms of dk_1', Eq. (30b) becomes

$$\sigma_{\text{ion}} = \frac{1}{2k_0}\frac{\hbar^2}{mK^3}\int d\mathbf{k}_1'\, d\mathbf{k}_2'\, \delta(E'-E) \times |A(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2')|^2, \tag{32}$$

where now $\mathbf{k}_1'$, $\mathbf{k}_2'$ range over all real values, with E' defined by the right side of Eq. (20).

Equation (32) is the desired expression for σ_{ion}. When the symmetry requirements of particle indistinguishability are ignored, e.g., when the second term on the right side of (14) is dropped in the definition of Ψ, it can be seen that[3]

$$A(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2') = \left(\frac{2m}{\hbar^2}\right)^3 \frac{e^{-3i\pi/4}}{2E^{\frac{1}{2}}}\left(\frac{E^{\frac{1}{2}}}{2\pi(2m/\hbar^2)^{\frac{1}{2}}}\right)^{5/2} \times T(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2'), \tag{33}$$

where T is the usual transition amplitude

$$T(i\to f) = \int \Psi_f^{(-)*} V_i \psi_i \tag{34}$$

from initial to final states. In this unsymmetrized case, therefore, realizing that the factor $\frac{1}{2}$ must be dropped because now the incident current density is only $\hbar k_0/m$, Eq. (32) takes the familiar form

$$\sigma_{\text{ion}} = \frac{m}{\hbar k_0}\frac{2\pi}{\hbar}\frac{1}{(2\pi)^6}\int d\mathbf{k}_1'\, d\mathbf{k}_2'\, \delta(E'-E) \times |T(\mathbf{n}\to\mathbf{k}_1', \mathbf{k}_2')|^2. \tag{35}$$

In the symmetrized case, where all terms in Eq. (14) are retained, one also can retain Eq. (33) in which event Eq. (35) again holds provided $\frac{1}{2}$ is restored.

III. PROOF OF MOMENTUM-TRANSFER THEOREM

With the foregoing results in hand, the desired momentum-transfer cross-section theorem can be derived. As in the simpler case of potential scattering[1]

$$-\int d\mathbf{r}\, (H\Psi)^* p_{1z}\Psi + \int d\mathbf{r}\, \Psi^* p_{1z} H\Psi = 0, \tag{36}$$

where $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ is the function defined by Eqs. (14) and (15); Eq. (11) defines H; $p_{1z} = (\hbar/i)\partial/\partial z_1$ is the z component of the momentum of particle 1; the z direction now is supposed to coincide with the incident direction $\mathbf{n}$; and $d\mathbf{r} \equiv d\mathbf{r}_1 d\mathbf{r}_2$ signifies integration over all $\mathbf{r}_1$, $\mathbf{r}_2$. Again as previously,[1] to keep the integrals in (36) convergent, the integration volume may at first be supposed to equal the interior of a six-dimensional sphere (in $\mathbf{r}_1$, $\mathbf{r}_2$ space) of finite though very large radius. Whatever the integration volume, Eq. (36) is true because Ψ satisfies Eq. (4).

Using (11), Eq. (36) becomes

$$\frac{-\hbar^2}{2m}\int d\mathbf{r}\left\{\Psi^*(\nabla_1^2+\nabla_2^2)\frac{\partial\Psi}{\partial z_1} - [(\nabla_1^2+\nabla_2^2)\Psi]^*\frac{\partial\Psi}{\partial z_1}\right\} + \int d\mathbf{r}\,\Psi^*\frac{\partial V}{\partial z_1}\Psi = 0, \tag{37}$$

with V given by (12). The next step is to substitute (14) into the first integral of Eq. (37), thereby obtaining 18 independent pairs of terms under the integral sign. Most of these pairs vanish, however. For example,

$$\int d\mathbf{r}\left\{e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0^*(r_2)\nabla_1^2\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2) - \left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)\right]\nabla_1^2 e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0^*(r_2)\right\} = 0$$

because

$$(\nabla_1^2 + k_0^2)e^{\pm ik_0\mathbf{n}\cdot\mathbf{r}_1} = 0. \tag{38}$$

Also, holding $\mathbf{r}_2$ fixed and employing Green's theorem in the three-dimensional $\mathbf{r}_1$ space, one sees that

$$\int d\mathbf{r}_1\, d\mathbf{r}_2\left\{e^{-ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0^*(r_1)\nabla_1^2\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2) - \left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)\right]\nabla_1^2 e^{-ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0^*(r_1)\right\} = 0$$

because $\phi_0(r_1)$ is exponentially decreasing as $r_1 \to \infty$; similarly, pairs of terms involving ∇_2^2 and $\phi_0(r_2)$ are seen to vanish after employing Green's theorem in $\mathbf{r}_2$ space.

In this fashion, Eq. (37) yields

$$\int d\mathbf{r}\left\{\Phi^*\nabla_1^2\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)-\left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)\right]\nabla_1^2\Phi^*\right\}$$
$$+\int d\mathbf{r}\left\{e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0^*(r_2)\nabla_1^2\frac{\partial\Phi}{\partial z_1}-\frac{\partial\Phi}{\partial z_1}\nabla_1^2e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0^*(r_2)\right\}$$
$$+\int d\mathbf{r}\left\{\Phi^*\nabla_2^2\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0(r_1)-\left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0(r_1)\right]\nabla_2^2\Phi^*\right\}$$
$$+\int d\mathbf{r}\left\{e^{-ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0^*(r_1)\nabla_2^2\frac{\partial\Phi}{\partial z_1}-\frac{\partial\Phi}{\partial z_1}\nabla_2^2e^{-ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_0^*(r_1)\right\}$$
$$+\int d\mathbf{r}\left\{\Phi^*(\nabla_1^2+\nabla_2^2)\frac{\partial\Phi}{\partial z_1}-\frac{\partial\Phi}{\partial z_1}(\nabla_1^2+\nabla_2^2)\Phi^*\right\}-\frac{2m}{\hbar^2}\int d\mathbf{r}\,\Psi^*\frac{\partial V}{\partial z_1}\Psi=0. \tag{39}$$

Reduction to Surface Integrals

Green's theorem in $\mathbf{r}_1$-space can be employed in the first integral of Eq. (39). Thus, using (16), this first term reduces to

$$\int d\mathbf{r}_2\int d\mathbf{S}_1\cdot\left\{\Phi^*\nabla_1\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)-\left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\phi_0(r_2)\right]\nabla_1\Phi^*\right\}$$
$$=\int d\mathbf{S}_1\cdot\left\{A_0^*(\mathbf{n}\to\mathbf{n}')\frac{e^{-ik_0r_1}}{r_1}\nabla_1\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}-\left[\frac{\partial}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_1}\right]\nabla_1A_0^*(\mathbf{n}\to\mathbf{n}')\frac{e^{-ik_0r_1}}{r_1}\right\}, \tag{40}$$

where $d\mathbf{S}_1=r_1^2\,d\mathbf{n}'$ is the surface element on the sphere at infinity in three-dimensional $\mathbf{r}_1$ space. Similarly, the second term in (39) reduces to

$$\int d\mathbf{S}_1\cdot\left\{e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\nabla_1\frac{\partial}{\partial z_1}A_0(\mathbf{n}\to\mathbf{n}')\frac{e^{ik_0r_1}}{r_1}-\left[\frac{\partial}{\partial z_1}A_0(\mathbf{n}\to\mathbf{n}')\frac{e^{ik_0r_1}}{r_1}\right]\nabla_1e^{-ik_0\mathbf{n}\cdot\mathbf{r}_1}\right\}. \tag{41}$$

Using Green's theorem in $\mathbf{r}_2$ space, the third integral in (39) becomes

$$\int d\mathbf{r}_1\int d\mathbf{S}_2\cdot\sum_j\left\{\phi_j^*(\mathbf{r}_1)A_j^*(\mathbf{n}')\frac{e^{-ik_jr_2}}{r_2}\frac{\partial\phi_0(r_1)}{\partial z_1}\times\nabla_2e^{ik_0\mathbf{n}\cdot\mathbf{r}_2}-\frac{\partial\phi_0(r_1)}{\partial z_1}e^{ik_0\mathbf{n}\cdot\mathbf{r}_2}\phi_j^*(\mathbf{r}_1)\nabla_2A_j^*(\mathbf{n}')\frac{e^{-ik_jr_2}}{r_2}\right\}, \tag{42}$$

where $d\mathbf{S}_2=r_2^2\,d\mathbf{n}'$. The quantity $A_j(\mathbf{n}')$ in (42) is identical with $A_j(\mathbf{n}\to\mathbf{n}')$ defined in (16) because

$$\lim_{\substack{r_2\to\infty\|\mathbf{n}'\\ r_2=\rho}}\int d\mathbf{r}_1\,\phi_j^*(\mathbf{r}_1)\Phi(\mathbf{r}_1,\mathbf{r}_2)$$
$$=\lim_{\substack{r_1\to\infty\|\mathbf{n}'\\ r_1=\rho}}\int d\mathbf{r}_2\,\phi_j^*(\mathbf{r}_2)\Phi(\mathbf{r}_2,\mathbf{r}_1)$$
$$=\lim_{\substack{r_1\to\infty\|\mathbf{n}'\\ r_1=\rho}}\int d\mathbf{r}_2\,\phi_j^*(\mathbf{r}_2)\Phi(\mathbf{r}_1,\mathbf{r}_2)=A_j(\mathbf{n}\to\mathbf{n}')\,e^{ik_j\rho}/\rho. \tag{43}$$

The first equality in (43) simply interchanges the labeling on $\mathbf{r}_1$, $\mathbf{r}_2$; the second equality makes use of (15). I now observe that when $k_j\neq k_0$ the integral in (42) oscillates infinitely rapidly at infinite r_2 and gives no net contribution when averaged over any small range of incident energies. Hence, the terms $k_j\neq k_0$ are inconsequential, and can be dropped[4] from (42). But the remaining term $k_j=k_0$ vanishes after integration over $\mathbf{r}_1$ because $\partial/\partial z_1$ has odd parity. Thus the expression (42), which equals the third integral in (39), vanishes. Similarly, the fourth integral in (39) vanishes.

The fifth integral in (39) is evaluated using Green's theorem in $\mathbf{r}_1$, $\mathbf{r}_2$ space. As explained in connection with Eq. (23), the surface elements at infinity in $\mathbf{r}_1$, $\mathbf{r}_2$ space are of the following different types: (a) $r_1\to\infty$, r_2 remains finite; (b) $r_2\to\infty$, r_1 remains finite; (c) both r_1, $r_2\to\infty$. Then, as in Eqs. (26)–(29), the contribution from surface elements of type (a) to the fifth integral in (39) is

[4] Similar terms are discarded, for the same reason, in the original derivation yielding the form of the many-particle current operator. See Ref. 1, especially Eq. (3.20).

$$\int d\mathbf{r}_2 \int d\mathbf{S}_1 \cdot \sum_{j,l} \left\{ \phi_j^*(\mathbf{r}_2) A_j^*(\mathbf{n}') \frac{e^{-ik_j r_1}}{r_1} \nabla_1 \phi_l(\mathbf{r}_2) \frac{\partial}{\partial z_1} A_l(\mathbf{n}') \frac{e^{ik_l r_1}}{r_1} - \left[\phi_l(\mathbf{r}_2) \frac{\partial}{\partial z_1} A_l(\mathbf{n}') \frac{e^{ik_l r_1}}{r_1} \right] \nabla_1 \phi_j^*(\mathbf{r}_2) A_j^*(\mathbf{n}') \frac{e^{-ik_j r_1}}{r_1} \right\} \tag{44a}$$

$$= \int d\mathbf{S}_1 \cdot \sum_j \left\{ A_j^*(\mathbf{n}') \frac{e^{-ik_j r_1}}{r_1} \nabla_1 \frac{\partial}{\partial z_1} A_j(\mathbf{n}') \frac{e^{ik_j r_1}}{r_1} - \left[\frac{\partial}{\partial z_1} A_j(\mathbf{n}') \frac{e^{ik_j r_1}}{r_1} \right] \nabla_1 A_j^*(\mathbf{n}') \frac{e^{-ik_j r_1}}{r_1} \right\}. \tag{44b}$$

The expression (44a) is simply the contribution to the fifth integral of (39) made by the terms involving ∇_1^2. The expression (44b) equals (44a) by virtue of the orthonormality of the $\phi_i(\mathbf{r}_2)$. Even if $\phi_i(\mathbf{r}_2)$ were not an orthogonal set, however, the terms $k_j \neq k_l$ in (44) would be inconsequential, just as in Eq. (42).

The contribution to the fifth integral of (39) from surface elements of type (b) (described in the preceding paragraph) is simply the contribution to that integral made by the terms involving ∇_2^2. This contribution, which also involves a double sum over j,l as in (44a) vanishes because: (i) terms $k_j \neq k_l$ are inconsequential; (ii) the fact that $\partial/\partial z_1$ has odd parity eliminates terms $k_j = k_l$. There remains the contribution to the fifth integral of (39) from surface elements of type (c). As in Eq. (23), this contribution is

$$\int dS \left\{ A^* \frac{e^{-iKr}}{r^{5/2}} \frac{\partial}{\partial r} \frac{\partial}{\partial z_1} A \frac{e^{iKr}}{r^{5/2}} - \left[\frac{\partial}{\partial z_1} A \frac{e^{iKr}}{r^{5/2}} \right] \frac{\partial}{\partial r} A^* \frac{e^{-iKr}}{r^{5/2}} \right\}, \tag{45}$$

where r is defined by Eq. (21); $A \equiv A(\mathbf{n} \rightarrow \mathbf{k}_1', \mathbf{k}_2')$ defined by Eqs. (19)–(22); dS is given by Eq. (31); and I have recognized that $\boldsymbol{\nu} \cdot \nabla = \boldsymbol{\nu}_1 \cdot \nabla_1 + \boldsymbol{\nu}_2 \cdot \nabla_2 = \partial/\partial r$ [$\boldsymbol{\nu}$ as in Eqs. (23) and (25), ∇ the six-dimensional gradient operator in $\mathbf{r}_1$, $\mathbf{r}_2$ space].

Surface Integrals Evaluated

The first five integrals in (39) have been reduced to (40), (41), (44b), and (45). I now shall evaluate these surface integrals. Using Eq. (12) of Ref. 1, one sees (just as in the case of potential scattering) that (40) and (41) together yield

$$\int d\mathbf{n}' \, 4\pi i k_0 (\mathbf{n} \cdot \mathbf{n}') \, \delta(\mathbf{n} - \mathbf{n}') [A_0^*(\mathbf{n} \rightarrow \mathbf{n}') - A_0(\mathbf{n} \rightarrow \mathbf{n}')] = 8\pi k_0 \, \mathrm{Im} \, A_0(\mathbf{n} \rightarrow \mathbf{n}). \tag{46}$$

The expression (44b) obviously reduces to

$$-\sum_j 2k_j^2 \int d\mathbf{n}' \, (\mathbf{n} \cdot \mathbf{n}') \, |A_j(\mathbf{n} \rightarrow \mathbf{n}')|^2. \tag{47}$$

Using (21), (22), and (31), the expression (45) is seen to equal

$$-2 \int \frac{dS}{r^5} |A|^2 K^2 \frac{z_1}{r} = -2 \int \frac{dS}{r^5} |A|^2 K^2 (\mathbf{n} \cdot \mathbf{n}_1') \frac{r_1}{r} = -2 \int \frac{dS}{r^5} |A|^2 K (\mathbf{n} \cdot \mathbf{k}_1'). \tag{48}$$

Thus, since (30a) can be put in the form (32), the right side of (48)—which equals (45)—can be expressed as

$$\frac{-2\hbar^2}{mK^3} \int d\mathbf{k}_1' \, d\mathbf{k}_2' \, \delta(E' - E) (\mathbf{n} \cdot \mathbf{k}_1') \, |A(\mathbf{n} \rightarrow \mathbf{k}_1', \mathbf{k}_2')|^2. \tag{49}$$

I next note that the definitions (22) imply the magnitudes k_1', k_2' of $\mathbf{k}_1'$, $\mathbf{k}_2'$ obey the relations

$$k_1'(q^{-1}) = k_2'(q), \qquad k_2'(q^{-1}) = k_1'(q). \tag{50}$$

Consequently, directly from the definition (19)

$$\lim_{\substack{r_1 \to \infty \| \mathbf{n}_2' \\ r_2 \to \infty \| \mathbf{n}_1' \\ r_2/r_1 = q^{-1}}} \Phi(\mathbf{r}_1, \mathbf{r}_2) = A(\mathbf{n} \rightarrow k_2'\mathbf{n}_2', k_1'\mathbf{n}_1') \frac{e^{iKr}}{r^{5/2}}, \tag{51}$$

where k_1', k_2' are $k_1'(q)$, $k_2'(q)$ of Eq. (22). Using (15), Eq. (51) can be rewritten as

$$\lim_{\substack{r_2 \to \infty \| \mathbf{n}_1' \\ r_1 \to \infty \| \mathbf{n}_2' \\ r_1/r_2 = q}} \Phi(\mathbf{r}_2, \mathbf{r}_1) = A(\mathbf{n} \rightarrow k_2'\mathbf{n}_2', k_1'\mathbf{n}_1') \frac{e^{iKr}}{r^{5/2}}. \tag{52}$$

Hence, because $\mathbf{r}_1$, $\mathbf{r}_2$ are just dummy variables in Eqs. (19) and (52), those equations imply

$$A(\mathbf{n} \rightarrow \mathbf{k}_1', \mathbf{k}_2') = A(\mathbf{n} \rightarrow \mathbf{k}_2', \mathbf{k}_1'). \tag{53}$$

Obviously, with indistinguishable electrons, the actual amplitude for ionization must obey a relation like (53). It seemed desirable to show that (53) indeed does follow from the definition of A, however; moreover, the fact that (53) can be proved supports the interpretation of the many-particle current operator (discussed in Sec. II), which interpretation led to the relations (30)–(32) between

$\sigma_{\rm ion}$ and A. Relabeling the dummy variables $\mathbf{k}_1'$ and $\mathbf{k}_2'$ in (49), and using (53), one sees that (45) equals

$$\frac{-2\hbar^2}{mK^3}\int d\mathbf{k}_2'\, d\mathbf{k}_1'\ \delta(E'-E)(\mathbf{n}\cdot\mathbf{k}_2')\ |A(\mathbf{n}\to\mathbf{k}_2',\mathbf{k}_1')|^2$$

$$=\frac{-\hbar^2}{mK^3}\int d\mathbf{k}_1'\, d\mathbf{k}_2'\ \delta(E'-E)(\mathbf{n}\cdot\mathbf{k}_1' + \mathbf{n}\cdot\mathbf{k}_2')\ |A(\mathbf{n}\to\mathbf{k}_1',\mathbf{k}_2')|^2. \quad (54)$$

Expressions for σ_d and σ

In the present e–H scattering problem, using (29) and (32), the total cross section is

$$\sigma = \sum_i \frac{k_i}{k_0}\int d\mathbf{n}'\ |A_i(\mathbf{n}\to\mathbf{n}')|^2 + \frac{1}{2k_0}\frac{\hbar^2}{mK^3} \times \int d\mathbf{k}_1'\, d\mathbf{k}_2'\ \delta(E'-E)\ |A(\mathbf{n}\to\mathbf{k}_1',\mathbf{k}_2')|^2. \quad (55)$$

Correspondingly, the definition (1) of the momentum-transfer cross section generalizes to

$$\sigma_d = \frac{1}{k_0}\sum_i \frac{k_i}{k_0}\int d\mathbf{n}'\ [k_0 - k_i(\mathbf{n}'\cdot\mathbf{n})]\ |A_i(\mathbf{n}\to\mathbf{n}')|^2 + \frac{1}{k_0}\frac{1}{2k_0}\frac{\hbar^2}{mK^3}\int d\mathbf{k}_1'\, d\mathbf{k}_2'\ \delta(E'-E)[k_0 - \mathbf{k}_1'\cdot\mathbf{n} - \mathbf{k}_2'\cdot\mathbf{n}]\ |A(\mathbf{n}\to\mathbf{k}_1',\mathbf{k}_2')|^2. \quad (56)$$

When multiplied by the incident velocity $\hbar k_0/m$, the first term on the right side of (56) obviously represents the rate (in units of the initial momentum $\hbar k_0$, to keep the dimensions of σ_d equal to length squared) with which momentum along the incident direction $\mathbf{n}$ is being transferred in excitation processes, including elastic scattering. Similarly, the last term in (56) obviously represents the momentum transfer by ionization, recognizing that when ionization occurs both electrons simultaneously carry away momentum.

The generalization of (7) to the present problem is

$$\sigma = (4\pi/k_0)\ \mathrm{Im}\ A_0(\mathbf{n}\to\mathbf{n}), \quad (57)$$

where σ is given by (55), and A_0 as always is the elastic forward scattering amplitude. In other words, although the particles are indistinguishable and A involves both ordinary and exchange amplitudes via (18), the optical theorem has exactly the same form as if the particles were distinguishable. If a proof of (57) is desired, it can be obtained by starting from

$$-\int d\mathbf{r}\ (H\Psi)^*\Psi + \int d\mathbf{r}\ \Psi^* H\Psi = 0 \quad (58)$$

instead of (36), and then reducing (58) to surface integrals along the lines employed earlier in this section.

Returning now to Eq. (39), the first five integrals in (39) have been reduced to the sum of (46), (47), and (54). Therefore, using (57), Eq. (39) yields

$$2k_0^2\sigma - \sum_i 2k_i^2\int d\mathbf{n}'\ (\mathbf{n}\cdot\mathbf{n}')\ |A_i(\mathbf{n}\to\mathbf{n}')|^2 \times \frac{-\hbar^2}{mK^3}\int d\mathbf{k}_1'\, d\mathbf{k}_2'\ \delta(E'-E)(\mathbf{k}_1'\cdot\mathbf{n}+\mathbf{k}_2'\cdot\mathbf{n}) \times |A(\mathbf{n}\to\mathbf{k}_1',\mathbf{k}_2')|^2 = \frac{2m}{\hbar^2}\int d\mathbf{r}\ \Psi^*\frac{\partial V}{\partial z_1}\Psi. \quad (59)$$

Using (55) to eliminate σ and dividing by $2k_0^2$, one sees that Eq. (59) implies Eq. (8).

There remains one point to be discussed before concluding this paper, namely, the effect of including Coulomb functions rather than plane waves in (14). Using Coulomb functions in (14) means the asymptotic form of (16) must be modified by inclusion of an extra factor,[5] proportional to $\exp(-i\eta_i \ln k_i r_1)$, where η_i is proportional to k_i^{-1}. Once this factor is included, the proof which has been given goes through essentially as in the plane-wave case, except that one must include derivatives of $\exp(-i\eta_i \ln k_i r_1)$ at infinity. But these derivatives, like the derivatives of r_1^{-1} itself, are of higher order in r_1^{-1} and so can be neglected at infinite r_1. This justifies the assertion, in Sec. I, that the momentum-transfer theorem should apply, e.g., to excitation of H^- by electrons. The argument in this paragraph also suggests the momentum-transfer theorem will remain valid in, e.g., ionization of H^- by electrons; for a more definitive statement, however, it is necessary to know how Eq. (19) must be modified when two electrons go out to infinity in the field of the proton[6] (fixed at the origin). The reader is reminded, moreover, of the remark in Sec. I that the right side of (8) apparently diverges for e–H^- collisions. In electron–ion collisions, therefore, Eq. (8) (whether or not it is essentially valid) is not likely to be very useful without imposition of suitable cutoffs.

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On the Proof and Uniqueness of Wulff's Construction of the Shape of Minimum Surface Free Energy

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The existing proof of Wulff's theorem shows that, among all *convex* bodies of fixed volume, the shape given by Wulff's construction has the least surface free energy. Here it is pointed out that the restriction to convexity is unnecessary: among *all* bodies of fixed volume, the shape given by Wulff's construction has (uniquely) the least surface free energy. Obvious generalizations are noted.

THE question of determining the equilibrium shape of a crystal whose specific surface free energy $\gamma(\hat{n})$ is a given function of unit surface normal, $\hat{n}$, was first posed by Gibbs.[1] The solution, in the form of a construction, was given by Wulff.[2]

Wulff's construction is as follows. From a fixed origin, erect vectors in the direction $\hat{n}$, of length proportional to $\gamma(\hat{n})$. For each such vector erect the plane normal to the vector which contains the end point of the vector. The largest closed body contained within all the members of this set of planes is then, according to Wulff, the equilibrium shape. That this shape must always be convex [for $\gamma(\hat{n}) \geq 0$] is clear; Wulff assumed in addition that the construction would always yield a convex polyhedron.

Wulff showed that the shape given by his construction provides a relative minimum in total surface free energy. Whether or not Wulff's construction actually gives an absolute minimum remained an open question until 1944, when Dinghas[3] showed that, among all convex polyhedra of fixed volume, the shape given by Wulff's construction (supposing that Wulff's construction does yield a polyhedron) has an absolute minimum in surface free energy. Dinghas' proof depends essentially upon the Brunn–Minkowski inequality, which, in the form quoted by him, states that for any two *convex* polyhedra A, B, having volumes $V(A)$ and $V(B)$, the volume of the Minkowski sum,[4] $A \times B$, of A and B satisfies

$$V(A \times B)^{\frac{1}{3}} \geq V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}},$$

with equality holding only when A and B are geometrically similar and similarly oriented.

More recently, Herring[5] has pointed out that Wulff's construction need not necessarily lead to polyhedra, but can give smoothly curved equilibrium shapes as well. The generalization of Dinghas' result for convex polyhedra to arbitrary convex bodies is immediate[5]; further, Herring conjectured that the restriction to convex bodies was not necessary, but offered no proof.

Mullins[6] has shown that, for the equivalent two-dimensional problem (the shape having least line free energy for fixed area), the equilibrium shape must be convex, but was not able to generalize the argument to three dimensions.

We wish to point out that the Brunn–Minkowski inequality is valid for nonconvex bodies as well[7,8]; the theorem has been proved in the following form:

For any two nonempty, closed, bounded sets of points A and B in Euclidean space of m dimensions the inequality

$$V(A \times B)^{1/m} \geq V(A)^{1/m} + V(B)^{1/m}$$

is valid; where, for sets of positive measure [i.e., $V(A)$, $V(B) > 0$], equality occurs when, and only when, A and B are geometrically similar and similarly oriented convex bodies.

Thus, citation of this stronger theorem in the Dinghas–Herring[3,5] proof is sufficient to prove that, among *all* bodies of fixed (finite) volume, the shape given by Wulff's construction has (uniquely) the least surface free energy.

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[2] G. Wulff, Z. Krist. **34**, 449 (1901).
[3] A. Dinghas, Z. Krist. **105**, 304 (1944).
[4] Let **oa** be a vector from some fixed origin to a point a belonging to the set A and, similarly, for **ob**, $b \in B$. Then, the set of all points c such that **oc** = **oa** + **ob** is the Minkowski sum set $A \times B$ of the sets A, B.
[5] C. Herring, in *Structure and Properties of Solid Surfaces*, edited by C. S. Smith and R. Gomer (The University of Chicago Press, Chicago, 1953).
[6] W. W. Mullins, J. Math. Phys. **3**, 754 (1962).
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It is of interest to point out that Wulff's solution is limited neither to the problem of minimizing surface free energy nor to three-dimensional Euclidean space. The general Brunn–Minkowski inequality[7,8] can be used to prove the following theorem for Euclidean space of m dimensions (E^m).

Given a positive function of direction in E^m, $f(\hat{n})$. Then, among all closed bodies of fixed m-dimensional volume, V_m, the [$(m-1)$-dimensional] surface integral of $f(\hat{n})$ is least when, and only when, the body is that given by Wulff's construction, using $(m-1)$ dimensional hyperplanes. Further, it can readily be shown that this minimum value is given by

$$m([f(\hat{n})+f(-\hat{n})]/w(\hat{n}))V_m,$$

where $f(\pm\hat{n})$ are the values of the function for opposed surface orientations and $w(\hat{n})$ is the width of the Wulff body of volume V_m in the direction $\hat{n}$.[9]

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Theorem on Nonradiative Electromagnetic and Gravitational Fields

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A vacuum electromagnetic field is considered which is (a) stationary below a hypersurface Σ intersecting all parametric lines of the time coordinate t and (b) nonradiative above Σ. It is proved that such a field is stationary also above Σ. The same theorem is proved to be valid for the pure gravitational field in general relativity and for the combined gravitational and electromagnetic field in the Einstein–Maxwell theory.

INTRODUCTION

It seems at present inevitable that approximation methods should be used to study gravitational radiation. One of the questions of primary interest is the discussion of the emission of gravitational radiation by a given nonstationary material system. To achieve this it is indispensable to formulate and use conditions which shall exclude incoming radiation. We shall not enter here in the rather delicate question of the possibility to find purely local forms of such a condition. Instead we shall consider only fields which have the following property: The field is stationary below a characteristic hypersurface Σ which will be supposed to coincide asymptotically with an elementary cone, open into the future, of the limiting Minkowski metric. Evidently there cannot be any incoming radiation in such a case. It might be useful to recall that this assumption has been used directly or indirectly in all recent work on gravitational radiation.[1–9]

For a clear formulation of the aim of this paper the following second remark will be needed. The gravitational field quantities which describe radiation at large distances from the sources are the terms of the curvature tensor which tend to zero as $1/r$ for $r \to \infty$. This result is contained implicitly already in the first discussion of gravitational radiation by Einstein[10] and has been confirmed directly by the recent work on the subject. A similar result holds, of course, also in the Maxwell theory (in special relativity): Electromagnetic radiation is described, at large distances from the sources, by the terms of the field tensor $F_{\mu\nu}$ which are proportional to $1/r$. We shall call these terms, for brevity, the radiative terms of the gravitational or the electromagnetic field.

The following theorem will now be proved. A field which is stationary below a characteristic hypersurface Σ and has no radiative terms above Σ will be stationary also above Σ. The proof will be complete for the Maxwell field in special relativity. But there will remain a certain lack of completeness in the case of the gravitational as well as of the combined gravitational-electromagnetic field. The exact nature of the mathematical difficulty which is still present will be clear later.

II. THE ELECTROMAGNETIC FIELD IN SPECIAL RELATIVITY

A. Proof Based on the Field Equations

Throughout this paper we shall assume that the material system which constitutes the sources of the field we are considering is contained, for all values of the time t, inside a cylinderlike tube around the t axis whose three-dimensional section is a sphere of radius a:

$$j^{\mu} = 0, \quad \text{for} \quad r > a \quad (\text{and every } t). \tag{1}$$

We shall not be interested in the field inside the region $r \leq a$.

We shall use Lorentz gauge,

$$\eta^{\mu\nu} A_{\mu,\nu} = 0, \tag{2a}$$

in which case the Maxwell equations are

$$\Box A_{\mu} = 0. \tag{2}$$

The retarded solution of this equation is given by the formula

$$\left.\begin{aligned} A_{\mu}(x^i, t) &= \int \frac{1}{R}\, j_{\mu}(X^i, T)\, d^3X; \\ T = t - R, &\qquad R^2 = (X^i - x^i)^2. \end{aligned}\right\} \tag{3}$$

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In the case of sources j_μ satisfying (1) it follows from (3) that the potential A_μ will be, in the region $r > a$, of the form

$$A_\mu = \sum_{n=1}^{\infty} \frac{1}{r^n} \cdot {}_nf_\mu(t - r, \xi^i); \tag{4}$$

$$r^2 = (x^i)^2, \qquad \xi^i = x^i/r. \tag{4a}$$

The expression (4) must of course satisfy Eqs. (2a) and (2).

It will be preferable to work with the electromagnetic field $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$: We shall in this way avoid any trivial "radiative" terms which might occur in A_μ. It is seen at once that $F_{\mu\nu}$ will be again of the form (4):

$$F_{\mu\nu} = \sum_{n=1}^{\infty} \frac{1}{r^n} \, {}_nF_{\mu\nu}(t - r, \xi^i). \tag{5}$$

Maxwell's first set of field equations now reads

$$\Box F_{\mu\nu} = 0. \tag{6}$$

There is also the second set of equations,

$$F_{[\mu\nu,\lambda]} = 0. \tag{7}$$

For the proof of our theorem it will be sufficient to use (5) and (6).

We shall now state exactly the assumptions on which the proof of our theorem will be based.

(a) The field is stationary below Σ:

$$F_{\mu\nu,4} = 0 \quad \text{or equivalently} \quad {}_nF_{\mu\nu,4} = 0 \quad \text{for all } n. \tag{8}$$

It is known from elementary considerations that a stationary electromagnetic field cannot have a term proportional to $1/r$. Hence we shall have, besides (8),

$${}_1F_{\mu\nu} = 0. \tag{8a}$$

(b) The field has no radiative term above Σ; i.e., we shall have the condition (8a) in the whole of space–time.

(c) The demand that Σ coincides asymptotically with an elementary cone opening into the future has the consequence that the equation of Σ will be of the asymptotic form

$$t - r + \cdots = \text{const} \tag{9}$$

the omitted terms being independent of t and of order r^λ with $\lambda \leq 0$.

For the proof of our theorem we shall need the following relation, derived by a straightforward computation:

$$\Box \frac{f(t - r, \xi^i)}{r^n} = -\frac{2(n-1)}{r^{n+1}} \frac{\partial f}{\partial t} + \frac{1}{r^{n+2}} \times \left\{ (\xi^k \xi^l - \delta^{kl}) \frac{\partial^2 f}{\partial \xi^k \partial \xi^l} + 2\xi^k \frac{\partial f}{\partial \xi^k} - n(n-1)f \right\}. \tag{10}$$

According to assumptions (a) and (b) the expression (5) for $F_{\mu\nu}$ will now start with the term $n = 2$. It follows then from (10) that the leading term of the left-hand side of Eq. (6) will be $-(2/r^3)\, {}_2F_{\mu\nu,4}$.

[When we consider a field with a radiative term, ${}_1F_{\mu\nu} \neq 0$, the left-hand side of the field equation (6) starts again with a term proportional to $1/r^3$, the term proportional to $1/r^2$ dropping out because of the factor $n - 1$. The physical meaning of this feature is that the field equations do not determine ${}_1F_{\mu\nu,4}$, the time dependence of the radiative part of the field being arbitrary. The second set of Maxwell's equations (7) imposes certain restrictions on the six quantities ${}_1F_{\mu\nu}$ with the well-known result: The leading parts of the electric as well as of the magnetic field are orthogonal to the radial direction as well as to each other and of equal strength; the necessary and sufficient condition for the nonexistence of electromagnetic radiation is thus found again to be ${}_1F_{\mu\nu} = 0$.]

Consequently, it follows from the field equation (6) that we shall have ${}_2F_{\mu\nu,4} = 0$ in the whole of space–time and not only below Σ as we assumed in (8).

Again by using (10) we find that the next term of the left-hand side of (6) is

$$\frac{1}{r^4} \left\{ -4\, {}_3F_{\mu\nu,4} + (\xi^k \xi^l - \delta^{kl}) \frac{\partial^2}{\partial \xi^k \partial \xi^l} \, {}_2F_{\mu\nu} + 2\xi^k \frac{\partial}{\partial \xi^k} \, {}_2F_{\mu\nu} - 2\, {}_2F_{\mu\nu} \right\}.$$

The coefficient of $1/r^4$ must vanish because of (6):

$${}_3F_{\mu\nu,4} = \frac{1}{4} \left\{ (\xi^k \xi^l - \delta^{kl}) \frac{\partial^2}{\partial \xi^k \partial \xi^l} + 2\xi^k \frac{\partial}{\partial \xi^k} - 2 \right\} {}_2F_{\mu\nu}.$$

Since ${}_2F_{\mu\nu}$ has been already found to be time independent, it follows from this last relation that ${}_3F_{\mu\nu,4}$ will be also time independent. But according to assumption (8) ${}_3F_{\mu\nu,4}$ vanishes below Σ. Hence, it will vanish in the whole space–time. By the same reasoning we find from the subsequent terms of equation (6) that ${}_4F_{\mu\nu,4}$, ${}_5F_{\mu\nu,4}$ and so on will vanish in the whole space–time. But this is then the theorem we wanted to prove.

It might be useful to remark that for this proof it is not necessary to use a characteristic hypersurface Σ. Exactly the same reasoning can be repeated with any hypersurface Σ', subject only to the condition that it intersects all the parametric lines of the coordinate t at points with values $t > -\infty$.

The general form of a nonradiative electromagnetic field, which is not supposed to satisfy the condition (8), can be easily determined by a similar reasoning. We firstly derive from the vanishing of the coefficient of $1/r^3$ in (6), as before, ${}_2F_{\mu\nu,4} = 0$. But now from the vanishing of the coefficient of $1/r^4$ in (6) we conclude only that ${}_3F_{\mu\nu,4}$ is time-independent, or equivalently ${}_3F_{\mu\nu,44} = 0$. Similarly, the coefficient of $1/r^5$ gives ${}_4F_{\mu\nu,444} = 0$; the general result is given by the condition

$$\partial^{n-1}({}_nF_{\mu\nu})/\partial t^{n-1} = 0. \tag{11}$$

B. Proof Based on the Properties of Shock Waves

It has not been possible to prove this theorem by a similar method for the gravitational field, when the exact field equations of general relativity are used. For this reason we shall now give a second proof of the theorem for the electromagnetic field. We shall then see later that it will be possible to apply this new method to the gravitational case too.

This proof will be based on the properties of shock waves. We therefore must start by enumerating the essential properties of electromagnetic shock waves. [See e.g. Ref. 11, where these properties are given in the form which will be useful to us here. Actually Stellmacher discusses the combined Einstein–Maxwell theory. To arrive at the formulae which are valid in the simple Maxwell theory (in special relativity) one has to put $g_{\mu\nu} = \eta_{\mu\nu}$ in the whole space-time.] For a more concrete reasoning we shall consider a shock wave of order 2; it will be seen later that the results at which we shall arrive are valid for shock waves of any order. Thus we shall now assume that on the hypersurface Σ there are discontinuities of the second derivatives of the electromagnetic potential A_μ. These discontinuities, normally denoted by $[A_{\mu,\rho\sigma}]$, are of the form

$$[A_{\mu,\rho\sigma}] = \varphi_\mu p_\rho p_\sigma, \tag{12}$$

p_ρ being the normal to the hypersurface Σ.

Further we shall assume that the shock wave is a genuine, physically meaningful one; i.e., we exclude trivial shock waves, the discontinuities of which can be eliminated by a gauge transformation. Σ must then be a characteristic hypersurface and p_ρ is a null vector at every point of Σ, $p^\rho p_\rho = 0$. The discussion will be simplified if we assume that any trivial discontinuities, which might be initially present, have been eliminated by an appropriate gauge transformation, so that only the genuine discontinuities remain. The quantity φ_μ entering in (12) will then behave like a vector. From the Maxwell equations we find for φ_μ firstly the local condition

$$\varphi_\mu p^\mu = 0. \tag{13}$$

In the case of a genuine shock wave φ_μ cannot be parallel to p_μ and therefore it will necessarily be a spacelike vector. Thus the necessary and sufficient condition for the existence of a genuine shock wave is

$$-\varphi_\mu \varphi^\mu \equiv J > 0. \tag{14}$$

The scalar J may be called the amplitude of the shock wave.

Besides the algebraic local condition (13) there are two differential or propagation conditions on φ_μ which also follow from the Maxwell equations. The first one contains the amplitude J and has the form of the continuity equation

$$(Jp^\mu)_{;\mu} = 0. \tag{15}$$

It is this equation which will allow us to arrive at a proof of the theorem. Since $J \neq 0$, we can write (15) in the form

$$(\ln J)_{,\mu} p^\mu + p^\mu{}_{;\mu} = 0. \tag{15a}$$

It will be convenient to use cartesian coordinates, in which case the covariant derivatives reduce to ordinary ones.

The equation of the hypersurface Σ being given by (9), we find for its normal vector p_μ or p^μ the asymptotic form

$$p_\mu = (-\xi^i + \cdots, 1), \quad p^\mu = (\xi^i + \cdots, 1), \tag{16}$$

the omitted terms being at least of order $1/r$. From the second of these equations we derive by an immediate computation

$$p^\mu{}_{,\mu} = 2/r + \cdots,$$

the omitted terms being at least of order $1/r^2$. To calculate the first term of Eq. (15a) we must remember that J is a function defined only on Σ. It will therefore be convenient to consider it as a function of x^i only, t being on Σ a function of x^i according to (9). It then follows that the leading part of the first term of Eq. (15a) is

$$(\ln J)_{,i}\xi^i = \partial(\ln J)/\partial r.$$

Thus Eq. (15a) reduces to

$$\partial(\ln J)/\partial r + 2/r + \cdots = 0. \tag{17a}$$

The integration of this equation gives

$$J = r^{-2} f(\theta, \varphi) + \cdots \tag{17}$$

with $f \neq 0$. From (17) and (14) we conclude that the discontinuity vector φ_μ of a genuine shock wave will necessarily have a nonvanishing part proportional to $1/r$.

From (12) and the definition of $F_{\mu\nu}$, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, we derive the relation

$$[F_{\mu\nu,\alpha}] = (\varphi_\nu p_\mu - \varphi_\mu p_\nu)p_\alpha. \tag{18}$$

This equation can be solved for φ_μ, if we introduce a vector n^μ satisfying the two conditions

$$\varphi_\mu n^\mu = 0, \qquad p_\mu n^\mu = 1.$$

Multiplication of (18) by $n^\mu n^\alpha$ gives

$$[F_{\mu\nu,\alpha}]n^\mu n^\alpha = \varphi_\nu. \tag{18a}$$

The meaning of this relation is evidently that, when a genuine shock wave of order 2 is propagated on Σ, the discontinuity of the first derivative of $F_{\mu\nu}$ will necessarily have a (nonvanishing) part proportional to $1/r$.

We shall only mention that in the case of a shock wave of order $n \neq 2$ we have again equations identical to (13), (14), and (15). The only difference is that φ_μ is now defined by the relation

$$[A_{\mu,\alpha_1\alpha_2\cdots\alpha_n}] = \varphi_\mu p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n}.$$

We find then, by repetition of the previous argument, that the derivatives of order $n - 1$ of $F_{\mu\nu}$ will have discontinuities which contain a (nonvanishing) part proportional to $1/r$.

Now according to our assumptions the relation (8a) is valid below as well as above Σ, the field starting in the whole region $r > a$ with a term proportional to $1/r^2$. The derivatives of $F_{\mu\nu}$ will then also start with terms proportional to $1/r^2$ and consequently there cannot be any part proportional to $1/r$ in the discontinuities of $F_{\mu\nu}$ or of its derivatives. But then it follows from (18a) that there cannot be any (genuine) shock wave on Σ; the same result will evidently be true for any other characteristic hypersurface of the same type situated above Σ.

A physicist might be willing to accept that the nonexistence of any shock wave on a characteristic hypersurface Σ, below which the field is stationary, means that the field will be stationary also above Σ. But such a conclusion is not justified mathematically: There are functions $f(t)$ which are constant in the region $t \leq t_0$ and nonconstant in $t > t_0$ and still have all derivatives $\partial^n f/\partial t^n$ continuous at $t = t_0$. For a complete clarification we shall have to use again the field equations. This is very easily done in the electromagnetic case. Indeed, we derived from the Maxwell equations the result that the general retarded nonradiative electromagnetic field will depend on t through the elementary functions t^λ, $\lambda = 0, 1, 2, \cdots$. But then it is seen at once that the derivative of order λ would be discontinuous on Σ. Therefore the nonexistence of a shock wave on Σ will really mean that the field must be stationary also above Σ.

THE GRAVITATIONAL FIELD

Gravitational shock waves have the same essential properties as the electromagnetic ones.[11] It is therefore possible to repeat the reasoning we have developed for the electromagnetic field also in the case of the gravitational field. We shall in this way prove that the same theorem is valid for the gravitational field too.

We shall consider only gravitational fields which satisfy the usual boundary conditions,

$$g_{\mu\nu} \to \eta_{\mu\nu} \quad \text{for} \quad r \to \infty.$$

Therefore there will exist in them characteristic hypersurfaces Σ having the properties we have demanded before; they will have again an equation of the asymptotic form (9). We now assume that on a given Σ of this type there is a genuine gravitational shock wave of order 2. The corresponding discontinuities of the second derivatives of $g_{\mu\nu}$ will be given by a formula similar to (12),

$$[g_{\mu\nu,\alpha\beta}] = \gamma_{\mu\nu} p_\alpha p_\beta, \tag{19}$$

p_α being again the normal to Σ, satisfying

$$p_\alpha p^\alpha = 0.$$

Further we shall assume that no apparent discontinuities exist on Σ: If such discontinuities would exist initially, we eliminate them by an appropriate coordinate transformation. Under these assumptions $\gamma_{\mu\nu}$ will behave as a tensor. Because of the Einstein equations, $R_{\mu\nu} = 0$, the tensor $\gamma_{\mu\nu}$ will satisfy the following local conditions:

$$\gamma_{\mu\nu} p^\nu = 0, \qquad \gamma_{\mu\nu} g^{\mu\nu} = 0. \tag{20}$$

We shall not enter into a more detailed description of the structure of $\gamma_{\mu\nu}$, but only state the result according to which the necessary and sufficient condition for the existence of a genuine gravitational shock wave is

$$\gamma_{\mu\nu}\gamma^{\mu\nu} \equiv K > 0. \tag{21}$$

For the amplitude K we get again from the Einstein equations the following propagation relation:

[11] F. K. Stellmacher, Math. Ann. **115**, 740 (1938).

$$(Kp^{\alpha})_{;\alpha} = 0. \tag{22}$$

Since we have assumed that $K > 0$, we can rewrite Eq. (22) in the form

$$(\ln K)_{,\alpha}p^{\alpha} + p^{\alpha}{}_{;\alpha} = 0. \tag{22a}$$

Now from the equation (9) of Σ we derive the asymptotic results:

$$p^{\alpha}{}_{;\alpha} = 2/r + \cdots,$$

$$(\ln K)_{,\alpha}p^{\alpha} = \partial(\ln K)/\partial r + \cdots.$$

Hence, Eq. (22a) will take the form

$$\partial(\ln K)/\partial r + 2/r + \cdots = 0.$$

This has exactly the form of Eq. (17a) and will therefore lead to a similar result:

$$K = r^{-2}f(\theta, \varphi) + \cdots \tag{23}$$

with $f(\theta, \varphi) \neq 0$. Going back to Eq. (21), we find finally that the discontinuity tensor $\gamma_{\mu\nu}$ of a genuine gravitational shock wave will necessarily contain a (nonvanishing) term proportional to $1/r$.

A similar result can be derived for the discontinuity of the Riemann tensor. In the case of a shock wave of order $n = 2$ we have the relation

$$2[R_{\mu\nu\alpha\beta}] = \gamma_{\mu\beta}p_{\nu}p_{\alpha} + \gamma_{\nu\alpha}p_{\mu}p_{\beta} - \gamma_{\nu\beta}p_{\mu}p_{\alpha} - \gamma_{\mu\alpha}p_{\nu}p_{\beta}. \tag{24}$$

Let n^{μ} be a vector with the properties

$$\gamma_{\mu\nu}n^{\nu} = 0, \qquad p_{\mu}n^{\mu} = 1.$$

Multiplying equation (24) by $n^{\nu}n^{\alpha}$ we find

$$2[R_{\mu\nu\alpha\beta}]n^{\nu}n^{\alpha} = \gamma_{\mu\beta}.$$

This relation shows that $[R_{\mu\nu\alpha\beta}]$ will necessarily contain a part proportional to $1/r$, if there is a genuine shock wave of the order $n = 2$ on Σ. We shall only mention that the same reasoning can be used in the case of a shock wave of order $n \neq 2$. The final result is that the derivatives of order $n - 2$ of $R_{\mu\nu\alpha\beta}$ will necessarily have discontinuities containing a part proportional to $1/r$.

We now recall that the two assumptions, under which the theorem we want to prove should be valid, are the following: The gravitational field is stationary below Σ and nonradiative above Σ. This means that the curvature tensor will have no part proportional to $1/r$ also in the region $r > a$. But then it follows immediately from our results that no genuine gravitational shock wave of any order can exist on Σ, or on any similar characteristic hypersurface situated above Σ.

The conclusion we have reached is of course weaker than the statement that the field should be stationary also above Σ. There is no doubt that in order to arrive at this statement one must get some additional help from the field equations. But we could not see how to obtain this result by using the exact Einstein equations.

It is of course quite easy to obtain this result with the help of the linearized Einstein equations of the first order. But such a simplified proof will no doubt be considered as not entirely satisfactory. It will therefore be sufficient if we indicate very briefly the main line of the reasoning without entering into the details of the calculations.

With the de Donder condition the field equation of first order is

$$\Box {}_{1}\mathfrak{g}^{\mu\nu} = 0. \tag{25}$$

The first-order term of $R_{\mu\nu\alpha\beta}$ is linear in ${}_{1}\mathfrak{g}^{\mu\nu}$ and will therefore satisfy the equation

$$\Box {}_{1}R_{\mu\nu\alpha\beta} = 0. \tag{25a}$$

We shall have to use the retarded solution of (25). This leads to relations similar to (4) and (5):

$${}_{1}\mathfrak{g}^{\mu\nu} = \sum_{n=1}^{\infty} \frac{1}{r^{n}}\, {}_{n}h^{\mu\nu}(t - r, \xi^{i}), \tag{26}$$

$${}_{1}R_{\mu\nu\alpha\beta} = \sum_{n=1}^{\infty} \frac{1}{r^{n}}\, {}_{n}h_{\mu\nu\alpha\beta}(t - r, \xi^{i}). \tag{26a}$$

Our assumptions are analogous to (8) and (8a):

$$\begin{aligned} {}_{n}h_{\mu\nu\alpha\beta,4} &= 0 \quad \text{below } \Sigma, \\ {}_{1}h_{\mu\nu\alpha\beta} &= 0 \quad \text{everywhere.} \end{aligned} \tag{27}$$

From these assumptions and the field equations (26a) we derive, with the help of the relation (10), exactly as in the electromagnetic case:

$${}_{2}h_{\mu\nu\alpha\beta,4} = {}_{3}h_{\mu\nu\alpha\beta,4} = \cdots = 0 \text{ everywhere;}$$

i.e., the theorem has been proved.

We shall add a last remark. The quantity ${}_{1}R_{\mu\nu\alpha\beta}$ is linear in the second derivatives of ${}_{1}\mathfrak{g}^{\mu\nu}$. It follows that in the case of a stationary field ${}_{1}R_{\mu\nu\alpha\beta}$ will start with the term proportional to $1/r^{3}$, i.e., we shall have also ${}_{2}h_{\mu\nu\alpha\beta} = 0$. This result can be derived from the equations (25a) combined with the Bianchi identity, ${}_{1}R_{\mu\nu[\alpha\beta,\gamma]} = 0$, and the symmetry relation ${}_{1}R_{\mu\nu\alpha\beta} = {}_{1}R_{\alpha\beta\mu\nu}$. From the same equations we could also derive the detailed structure of ${}_{3}h_{\mu\nu\alpha\beta}$. This corresponds to the fact that in the case of a stationary electromagnetic field one can derive from the two sets of Maxwell's equations firstly the vanishing of the term of order $1/r$ and then the detailed structure of the term of order $1/r^2$, i.e., the relations

$$_1F_{\mu\nu} = 0; \qquad _2F_{i4} = \text{const}\, \xi^i, \qquad _2F_{ik} = 0.$$

We shall not give any details as we feel that this kind of calculation is of little interest.

Though we have not found a fully satisfactory mathematical proof, it seems that there can be no doubt that the following theorem is valid: A gravitational field which is stationary below a hypersurface Σ and has no radiative part above Σ will be stationary everywhere. Or, equivalently, any gravitational field which is stationary below Σ and (essentially) nonstationary above Σ will necessarily contain emitted gravitational radiation. We recall that an equivalent conclusion is implicitly contained in the results of the discussion of the radiation problem obtained by Newman and Unti.[6] The same conclusion should of course follow also from the calculations according to the method of Bondi,[3] when it will be possible to develop them to a higher approximation.

THE COMBINED GRAVITATIONAL AND ELECTROMAGNETIC FIELD

We mention briefly that results similar to those obtained for the gravitational field will be valid also in the Einstein–Maxwell theory. The reason is that in this theory too there is a propagation relation having the same form as before[11]:

$$\{(K + \kappa J)p^\alpha\}_{;\alpha} = 0.$$

K and J are the amplitudes defined by (21) and (14) and κ is the gravitational constant. The important feature is that the quantity $K + \kappa J$ is again positive-definite and will be necessarily nonzero for any genuine shock wave. The rest of the reasoning is essentially identical to that given for the electromagnetic or the gravitational field. The proof is again incomplete as in the gravitational case.

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Backward Superconducting Switching*

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A discussion is given of the initial speeds at which superconducting material changes state according to the London electrody namics. These transitions are taken to occur in the form of phase boundary motions. Phase changes from the normal to the superconducting state and vice versa are considered for the cases in which the external magnetic field is radiated to the boundary of the superconducting material. A distinct difference is found in transition rates depending on whether the transition is from the superconducting or from the normal state. In another case considered, the transition from normal to super is studied when the superconducting material is bounded by a good conductor. In all cases, constant critical field is taken as the switching criterion. The mathematical treatment involves the approximate solution of free boundary problems and mixed hyperbolic–parabolic boundary value problems.

1. INTRODUCTION

THE use of superconducting elements in computer switching devices requires knowledge of the dynamic behavior of superconducting materials as they change from the superconducting state to the normally conducting state. Although there has been an exceedingly fruitful development of the understanding of the fundamentals of superconductivity on a microscopic level in recent years, a time-dependent theory has not yet evolved. There is, however, a phenomenological model, that of London, which is known to be reasonably appropriate under certain limiting size and impurity requirements, and which can provide an electrodynamics theory. In fact, it appears to be customary to expect any theory of superconductivity to reduce, at least in form, to London theory when the ratio of the dimensions of the super conductor to an intrinsic parameter of the material, the penetration depth, becomes large. Ittner[1] has suggested a method of retaining the simplicity of the London formulation while taking into account the necessary quantitative alterations that are known to occur as the ratio mentioned above becomes smaller. With this in mind, this paper treats some cases of dynamic super conductivity in the London electrodynamics. The model taken for the transition is that of a phase change and a phase boundary motion. It is understood, of course, that the phase boundary will, in reality, be a transition region, perhaps of the order of the coherence length. Thus, the moving interface in the problems discussed here will be understood as a limiting case of this region.

The transition speeds that occur when superconducting elements are transformed from one state to another by altering the external magnetic field have been calculated from the phase change model for the case in which a superconductor is switched to a normal conductor. The transition rates for bulk materials with zero penetration depth (when the ratio mentioned above is infinite) have been given both for the isothermal case[2] and for the case in which eddy-current heating and latent heat are taken into account[3,4]. The effect of nonzero penetration depth on transition in bulk conductors (the ratio is large but finite) and in finite-width films has also been treated[5,6] under isothermal conditions. The present report includes consideration of what might be called the backward switching problem, the isothermal change from the normal state to the superconducting state. The case taken up will be that of a half-space of normal material at one edge of which a subcritical magnetic field is applicable.

The objective of the present study is to understand how, within the London theory, the superconducting region grows and expels magnetic flux from the normal region. Faber[7] has experimented with initially normally conducting but supercooled rods, and has found that the superconducting state may be induced by locally reducing the magnetic field strength at a spot on the surface of the rod. In the experiments a linear superconducting filament was found to travel down the length of the rod, remaining on the surface. Next a sheath formed circumferentially on the surface and, finally, after the

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[1] W. B. Ittner, III, Phys. Rev. 119, 1591 (1960).

[2] A. B. Pippard, Phil. Mag. 41, 243 (1950).
[3] W. B. Ittner, III, Phys. Rev. Suppl. 111, 148 (1958).
[4] A. W. Duijvestijn, IBM J. Res. Develop. 3, 2, 132 (1959).
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[6] W. Liniger, J. Math. Phys. 3, 3, 578 (1962).
[7] T. E. Faber, Proc. Roy. Soc. (London) A223, 174 (1954).

sheath closed about the surface, a very slow penetration began towards the interior. The half-plane problem studied here may be considered to represent only the final stage. The rate at which the superconducting region moved into the normal portion of the rods was such that Faber could explain it only on the basis of an interstitial leaking outward of the flux. For any real material this is undoubtedly a part of the explanation. The question posed here, however, is what London theory predicts about the casting out of the magnetic field from the normal region. The rate of transition for this backward switching will be compared with the rate of transition from the superconducting to the normal state. Because only the half-plane is considered, the question of how long it takes to completely cast out flux from a finite body does not occur. In fact, the results obtained here show only how the transition process begins in each case.

The analysis quickly indicates that the external magnetic environment must be examined to understand backward switching. Section 2 of this paper, in which the equations and boundary conditions are derived, discusses this point. In Sec. 3, a particular external magnetic field, a magnetic signal propagated through a vacuum to the superconducting material, is considered. The transition is described in its initial phase. In Sec. 4, as a comparison, a superconducting half-space is switched to the normal state by the same sort of external driving field. Both of these analyses differ from previous transition studies in that a nonzero rise-time for the driving signal is allowed. In Sec. 5, a superconducting to normal transition is studied in the case that the external region is a good conductor. There have been several recent considerations of the behavior of superconductors in the neighborhood of highly conducting, but not superconducting, metals.[8,9]

Section 6 of the paper contains the results from some numerical examples that have been carried out. The idea has been only to exhibit a comparison of initial transition speeds for the various cases considered and not to make a comprehensive numerical study. It is shown that for bulk materials the transition produced by a signal radiated through a vacuum to the superconducting material depends on the rise-time of the signal. For very fast rise-times, the penetration depth has little influence on the interface propagation rate in forward or backward switching. As the signal rise-times become slower, penetration effects take hold. It appears that the backward switching process becomes slower than forward switching. In fact, the speed of the backward phase boundary decreases with the square root of the rise-time of the input. On the other hand, the forward motion appears to approach in the limit a simple function of the penetration depth. This indicates a rather definite difference in behavior of the two, which is discussed in Sec. 6.

It is difficult to compare the backward transition speed in the case in which the external region is a good (nonsuperconducting) conductor with previous results for the forward transition under a step-function driving field. As is explained in Sec. 5, other conditions are required for the backward switching to begin than in the forward case, and, therefore, additional parameters are presented in the calculation. Included in Sec. 6 is an indication of the speeds achieved in this case and in its asymptotic stage.

It should be borne in mind that the authors of this paper are aware of the limitations of London theory and its electrodynamics. One important limitation is its neglect of nonlinear effects which lead to a dependence of the penetration depth on the field as described, for example, by the Ginzburg–Landau[10] equations. On the other hand, since there are no time-dependent Ginzburg–Landau equations at present, this paper has been done in the spirit of providing some insight into the mathematics of the London model as a limiting case of other, more inclusive, physical models.

Finally, it should be remarked that it is difficult to compare the results and proposals of Nethercott.[11] The present calculations rely heavily on the phase boundary model, whereas Nethercott's ideas are to be associated with transition regions.

2. FORMULATION OF THE PROBLEM

The formulation of the problem follows closely on that given in Ref. 4. Consider a half-space, $x > 0$, of superconducting material which for all time is held at a subcritical temperature. For $x < 0$, let there be a half space of nonsuperconducting material (see Fig. 1). All magnetic fields will be taken to be perpendicular to the plane of the paper such that there are current and electric field components in the positive y direction only (see Fig. 1). Furthermore, H, the magnetic field, will be allowed to assume only positive values.

The magnetic field in the whole space, for $T < 0$, is assumed to have attained a steady initial distribu-

[8] L. N. Cooper, Phys. Rev. Letters **6**, 689 (1961).
[9] R. H. Parmenter, Phys. Rev. **118**, 1173 (1960).
[10] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).
[11] A. H. Nethercott, Jr., Phys. Rev. Letters **7**, 226 (1961).

tion. If this is such that at $x = 0$ the field is below the critical value H_c, the region $x > 0$ will be superconducting and there will be a field, $H^+(x, 0)$, in the region (essentially an exponentially decaying field); if it is above, the half-space will be normal and the field, $H^+(x, 0)$, is a constant. At $T = 0$ the external field is altered so that the plane $x = 0$ eventually or instantaneously has $H(0, T) = H_c$. A transition front then moves into the right half-space, and it is the initial behavior of this front that will be studied in the sections that follow. This initial behavior depends on $H^+(x, 0)$, H_c, the external field ($x < 0$), and the material constants. One of the interesting observations is that it is not always sufficient to specify only the value of the magnetic field along $x = 0$, but it is also necessary to give information on the derivative, $H_x(0, T)$, or on the electric field, $E(0, T)$, along this line.

With reference to the equations given in London's book[12] and in Ref. 4, it is possible to set out a boundary-value problem for the switching process described above. It is convenient to normalize the actual dimensional coordinate x with respect to some scale of distance, x_0. and also to normalize the time scale by a factor, $(x_0^2/c^2)4\pi\sigma^N$, so that the dimensionless variables are $z = x/x_0$, $t = Tc^2/4\pi x_0^2\sigma^N$. c is the velocity of light and σ^N is the Ohmic conductivity in the normal region.

The differential equations that hold for the magnetic field in each of the three types of regions are as follows (+, −, 0 will be consistently taken to refer to these regions): superconducting region in the normal state,

$$H^+_{zz} = H^+_t + v_0^{-2}H^+_{tt}; \tag{2.1}$$

superconducting region,

$$H^-_{zz} = \alpha H^- + \beta H^-_t + v_0^{-2}H^-_{tt}; \tag{2.2}$$

nonsuperconducting (external) region,

$$H^0_{zz} = \beta^0 H^0_t + v_0^{-2}H^0_{tt}. \tag{2.3}$$

Here $\beta = \sigma^s/\sigma^N$ and $\beta^0 = \sigma^0/\sigma^N$, where σ^s, σ^0 are the Ohmic conductivities in the super and external regions, respectively. $v_0 = x_0(4\pi\sigma^N)c^{-1}$ is the dimensionless wave velocity in the external region under the present normalization.

These are derived by using the Maxwell and London equations. $\alpha = 4\pi x_0^2/\Lambda c^2$ is the square of the ratio of the arbitrary distance scale and the London penetration depth.

[12] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1950), Vol. I.

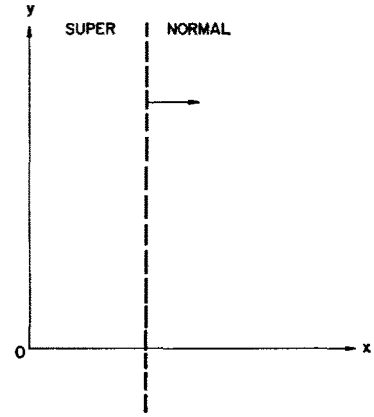

Fig. 1. The moving supernormal boundary in the physical space.

Actually Eqs. (2.1), (2.2), and (2.3) will take on special forms for the particular problems studied below. The term resulting from the displacement current will be neglected in regions in which there is a large conductivity. Thus, Eqs. (2.1) and (2.2) become

$$H^+_{zz} = H^+_t, \tag{2.1a}$$

$$H^-_{zz} = \alpha H^- + \beta H^-_t. \tag{2.2a}$$

Equation (2.3) will become

$$H^0_{zz} = \beta^0 H^0_t \tag{2.3a}$$

when the region exterior to the superconducting material has high conductivity (Sec. 5). However, it will reduce to

$$H^0_{zz} = v_0^{-2}H^0_{tt} \tag{2.3b}$$

when the exterior region is taken to be a vacuum (Secs. 3 and 4).

The free boundary, the interface between normal and super regions will be given by $x = \xi(T)$, which becomes $\zeta(t) = \xi(T)/x_0$ in the dimensionless system. Across the free boundary the magnetic field is continuous and attains the critical value, H_c. (The actual switching criterion is thought to be somewhere between a critical field and critical current criterion. We adopt here the simplest critical field hypothesis.) The total current is not continuous however, across this interface nor is it continuous across the boundary $z = 0$.

This is due, of course, to the presence of the supercurrent in the superconducting region. This discontinuity in the total current can be expressed as a single condition for the case in which the boundary moves into a half-infinite normal region (Fig. 2). This also represents the continuity of the electric field.

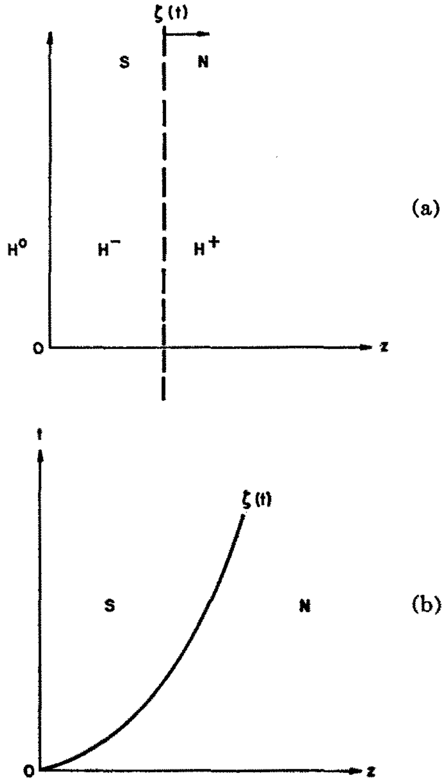

FIG. 2. (a) Magnetic fields and free boundary in physical space. (b) Moving boundary in z–t space.

$$\beta\partial H^{+}(\zeta(t), t)/\partial z - \partial H^{-}(\zeta(t), t)/\partial z = -\beta v_0 E^{0}(0, t) - \frac{\partial H^{-}}{\partial z}(0, t) - \alpha \int_0^{\zeta(t)} H^{-}(z, t)\, dz. \tag{2.4}$$

For the case in which the boundary moves into a half-infinite superconducting region (Fig. 3), the current discontinuity at $z = \zeta(t)$ takes the form

$$\beta \frac{\partial H^{+}}{\partial z}(\zeta(t), t) - \frac{\partial H^{-}}{\partial z}(\zeta(t), t) = \alpha \int_{\zeta(t)}^{\infty} H^{-}(\gamma, t)\, d\gamma. \tag{2.5}$$

Observe that because of the finite extent of the superconducting region in the first case [Eq. (2.4)], it is necessary to have information on both boundaries of the superconducting region. This occurs because the magnetic field is defined as an integral over the superconducting current density by the London equations. (In our notation the London equations are

$$\frac{\partial j_s}{\partial t} = \frac{x_0 v_0}{\Lambda c} E^{-}, \frac{\partial j_s}{\partial z} = -\frac{x_0}{\Lambda c} H^{-},$$

where j_s is the supercurrent; Λ is London's material constant.) In addition, because the two London relations are not identical in finite superconducting regions, Eq. (2.4) is not sufficient and must be augmented by the relation

$$-\partial H^{-}/\partial z(0, t) - \beta v_0 E^{0}(0, t) = v_0 \alpha \int_0^{t} E^{0}(0, \gamma)\, d\gamma. \tag{2.6}$$

This is applicable only in the case of a normal to superconducting transition when the superconducting region is finite.

Equations (2.4) and (2.5) provide the boundary conditions on the moving interface and Eq. (2.6) a supplementary one on the stationary interface, $z = 0$. The other boundary and initial conditions are more easily stated in terms of the particular problems and are given in the succeeding sections.

In Ref. 5, a discussion was given of the behavior of the Eqs. (2.1a) and (2.2a) and the interface condition when $\alpha \to \infty$. This would be the case of zero penetration distance, and it would seem as though $H^{-} \to 0$. It was shown, however, that for the super to normal transition, a special interface condition holds when $\alpha \to \infty$, namely:

$$\partial H^{+}(\zeta, t)/\partial z = -H_c\, d\zeta/dt. \tag{2.7}$$

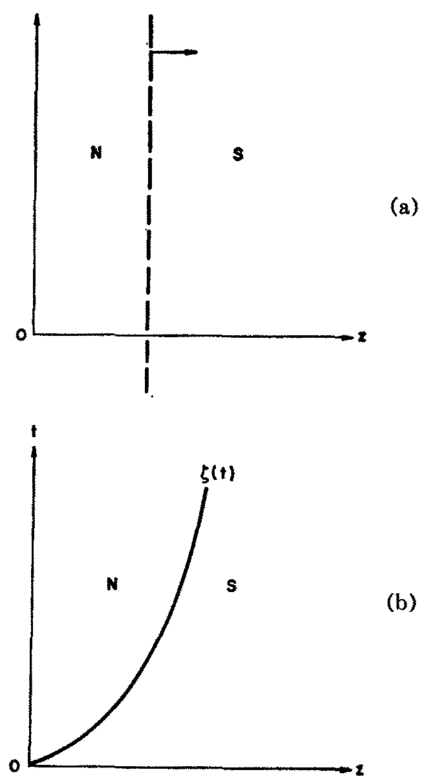

FIG. 3. (a) Forward switching in physical space. (b) Forward switching in z–t space.

Observe that $\partial H^+/\partial z < 0$ and thus $d\zeta/dt > 0$ for this super-to-normal case. In a similar manner, for a normal-to-super transition, one may obtain the same interface condition for $\alpha \to \infty$. But here $\partial H^+/\partial z$ must be > 0. Thus, $H^- \to 0$ does not seem to be an appropriate approximation even for very large α for the "backward" transition since it implies a negative speed for the phase boundary.

3. BACKWARD SWITCHING BY A PROPAGATING MAGNETIC WAVE

The special case to be considered here is represented in Figs. 4(a) and 4(b). The region $z < 0$ is a vacuum so that Eq. (2.3b) applies. For $t < 0$, there is a constant field H_e everywhere in the entire space where $H_e > H_c$. For $t \geq 0$, a source of magnetic field, located at $z = -l$, provides a signal $H^0(-l, t) = h^*(t) \leq H_e$. At $t = \tau^*$, $h^*(\tau^*) = H_c$. The signal is propagated through the distance l to $z=0$ so that at $t=l/v_0$ the value of H on $z=0$ begins to decrease from H_e toward H_c which it reaches at a certain time τ. Let the value of H on $z = 0$ be $w^*(t)$ so that $w^*(\tau) = H_c$. At time τ, a superconducting front forms on $z = 0$ and begins to propagate to the right. As the new region is formed, the field in it is governed by Eq. (2.2a) while in the normal region, $z > 0$ for $t < \tau$ and $z > \zeta(t)$ for $t > \tau$, Eq. (2.1a) is to be applied. The boundary value problem may be summarized here, but it is first convenient to slightly alter the variables $H^0(z, t)$ and $H^+(z, t)$. Let

$$u = [H^0(z, t) - H_e]/H_c, \qquad (3.1)$$

$$v = [H^+(z, t) - H_e]/H_c. \qquad (3.2)$$

Then for all $t > 0$, $-l < z < 0$,

$$u_{zz} = v_0^{-2} u_{tt} \qquad (3.3)$$

$$u(-l, t) = \frac{h^*(t) - H_e}{H_c} \equiv h(t), \quad h(\tau^*) = \frac{H_c - H_e}{H_c}, \qquad (3.4)$$

$$u(0, t) = \frac{w^*(t) - H_e}{H_c} \equiv w(t), \quad w(\tau) = \frac{H_c - H_e}{H_c}, \qquad (3.5)$$

$$w(0) = 0, \qquad (3.6)$$

$$u(z, 0) = u_t(z, 0) = 0. \qquad (3.7)$$

For $\tau > t > l/v_0$, $z > 0$

$$v_{zz} = v_t, \qquad (3.8)$$

$$v(z, l/v_0) = 0, \qquad (3.9)$$

$$v(0, t) = w(t), \qquad (3.10)$$

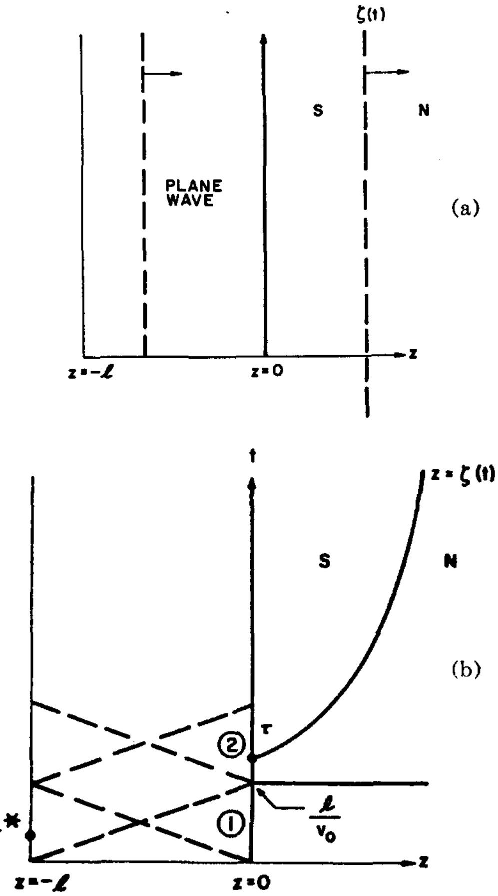

FIG. 4. (a) Backward switching by a propagating plane magnetic wave (physical space). (b) Backward switching by a propagating plane magnetic wave (z–t space).

$$v(z, t) \to 0, \qquad z \to \infty. \qquad (3.11)$$

For $t > \tau$, $z > \zeta(t)$

$$v_{zz} = v_t, \qquad (3.12)$$

$$v[\zeta(t), t] = (H_c - H_e)/H_c. \qquad (3.13)$$

$v(z, \tau)$ is prescribed by the solution of the problem for v in the region

$$\tau > t > l/v_0, \qquad z > 0.$$

For $t > \tau$, $0 < z < \zeta(t)$

$$H^-_{zz} = \alpha H^- + \beta H^-_t, \qquad (3.14)$$

or if

$$q(z, t) = \exp[\alpha\beta^{-1}(t - \tau)]H^-(z, t)/H_c, \qquad (3.15)$$

$$q_{zz} = \beta q_t, \qquad (3.16)$$

$$q(0, t) = \exp[\alpha\beta^{-1}(t - \tau)]w^*(t)/H_c, \qquad (3.17)$$

$$q(0, \tau) = 1, \qquad (3.18)$$

$$q(\zeta(t), t) = \exp[\alpha\beta^{-1}(t-\tau)], \tag{3.19}$$

$$\zeta(\tau) = 0. \tag{3.20}$$

As was pointed out in the previous section, to these equations must be added two other conditions (2.4) and (2.6), which now take the form

$$\beta \frac{\partial v}{\partial z}(\zeta(t), t) - \exp[-\alpha\beta^{-1}(t-\tau)]\frac{\partial q}{\partial z}(\zeta(t), t)$$

$$= -v_0\beta e^0(0, t) - \exp[-\alpha\beta^{-1}(t-\tau)]\frac{\partial q}{\partial z}(0, t)$$

$$-\alpha\int_0^{\zeta} \exp[-\alpha\beta^{-1}(t-\tau)]q(\gamma, t)\,d\gamma, \tag{3.21}$$

$$-\exp[-\alpha\beta^{-1}(t-\tau)]\partial q(0, t)/\partial z - \beta v_0 e^0(0, t)$$

$$= v_0\alpha\int_0^t e^0(0, \gamma)\,d\gamma, \tag{3.22}$$

where $t > \tau$ and $e^0 = E^0/H_c$.

This rather large set of equations and conditions may be broken up into the solution of three problems:

(1) For $t < l/v_0$, a wave propagates from $z = -l$ to $z = 0$ and, obviously, one will find no disturbance in the region $z > 0$.

(2) For $\tau > t > l/v_0$, wave motion carries the decreasing magnetic field to $z = 0$. The value of the field on $z = 0$, $w(t)$, decreases, and the magnetic and electric fields penetrate into the region $z > 0$.

(3) For $t > \tau$, since the critical field value is obtained at $t = \tau$, a superconducting front begins and the region $z > 0$ is devided into two sectors.

Before setting out on this program it is convenient to note that the electric field is given, as one approaches $z = 0$ from $z < 0$, by the expression

$$e^0(0^-, t) = e^0(0^-, 0) - v_0\int_0^t \frac{\partial u}{\partial z}(0^-, \gamma)\,d\gamma. \tag{3.23}$$

$e^0(0^-, 0)$ is equal to zero in the problem at hand.

(1) Now consider the solution of Eq. (3.3) subject to conditions (3.4), (3.5), (3.6), (3.7). It is most convenient to divide the region $-l < z < 0$, $t > 0$ into the sectors indicated by wavefront motion (i.e., the characteristics of the equation) and, in fact, the solution is only required in those sectors that border on $z = 0$. In the region labeled 1, Fig. 4, which is characterized by the inequalities $z < 0$, $(z - v_0t) \le l$, $(z + v_0t) \ge 0$,

$$u = w[(z + v_0t)/v_0]. \tag{3.24}$$

In region 2, characterized by $z < 0$, $(z - v_0t) \le 2l$, $(z + v_0t) \ge l$,

$$u = w\left(\frac{z+v_0t}{v_0}\right) - h\left(\frac{z+v_0t-l}{v_0}\right) + h\left(\frac{-z+v_0t-l}{v_0}\right). \tag{3.25}$$

At this point, as matter of convenience, it will be assumed that $2l/v_0 > \tau > l/v_0$. Solutions of the form exhibited for regions 1 and 2 can, of course, be written down quite simply for any such region.

(2) Turning to (3.8), a general solution for the equation may be written in the form

$$v(z, t) = \frac{z}{2\pi^{\frac{1}{2}}}\int_0^t \frac{\exp[-z^2/4(t-\sigma)]}{(t-\sigma)^{\frac{3}{2}}}w(\sigma)\,d\sigma. \tag{3.26}$$

Then, using Maxwell's equations,

$$e^+(0, t) = \frac{1}{v_0\pi^{\frac{1}{2}}}\int_0^t \frac{\partial w(\sigma)}{\partial\sigma}\frac{1}{(t-\sigma)^{\frac{1}{2}}}\,d\sigma. \tag{3.27}$$

Imposing the continuity of electric field condition $0 \le t \le l/v_0$, we get

$$-w(t) = \frac{1}{v_0\pi^{\frac{1}{2}}}\int_0^t \frac{\partial w(\sigma)}{\partial\sigma}\frac{1}{(t-\sigma)^{\frac{1}{2}}}\,d\sigma. \tag{3.28}$$

This integral equation has only the trivial solution $w \equiv 0$ and thus $v \equiv 0$ here as it must be. However, for $l/v_0 \le t \le \tau$, the integral equation for $w(t)$ is

$$2h\left(t - \frac{l}{v_0}\right) - w(t) = \frac{1}{v_0\pi^{\frac{1}{2}}}\int_0^t \frac{1}{(t-\sigma)^{\frac{1}{2}}}\frac{\partial w(\sigma)}{\partial\sigma}\,d\sigma. \tag{3.29}$$

The equation may be solved for $w(t)$ and yields for $l/v_0 \le t \le \tau$,

$$w(t) = 2v_0\int_0^t h\left(t - \frac{l}{v_0} - \sigma\right)\times\left[\frac{1}{(\pi\sigma)^{\frac{1}{2}}} - v_0\exp(v_0^2\sigma)\operatorname{erfc}(v_0\sigma^{\frac{1}{2}})\right]d\sigma, \tag{3.30}$$

where $h(t) \equiv 0$, $t < 0$.

The time τ may now be defined since it satisfies the condition

$$\frac{H_o - H_c}{H_o} = 2v_0\int_0^{\tau} h\left(\tau - \frac{l}{v_0} - \sigma\right)\times\left[\frac{1}{(\pi\sigma)^{\frac{1}{2}}} - v_0\exp(v_0^2\sigma)\operatorname{erfc}(v_0\sigma^{\frac{1}{2}})\right]d\sigma. \tag{3.31}$$

The solution in the region $z > 0$, $l/v_0 < t < \tau$ would now be obtained by inserting (3.30) into (3.26).

(3) For $2l/v_0 > t > \tau$, use must be made of the solution (3.25) and appropriate solutions for (3.8)

and (3.16). Now (3.21) and (3.22) must be imposed. However, the difficulty lies in the fact that the boundary $\zeta(t)$ is unknown. An exact solution in the superconducting and normal regions, $z > 0$, cannot be written down. Thus, the following plan has been adopted: General expressions are written for the solutions in the two regions in terms of undetermined singularity distributions along the infinite z axis. Then the distribution, as well as the unknown boundary $\zeta(t)$ are expanded in ascending series in powers of $(t - \tau)$. In this manner boundary conditions may be satisfied during the starting phase of the switching process. Letting $(t - \tau)$ now be denoted by t', the solutions have the form

$$q(z, t) = \beta^{\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{m(\sigma) \exp[-(z - \sigma)^2\beta/4t']}{2(\pi t')^{\frac{1}{2}}} d\sigma, \tag{3.32}$$

$$v(z, t) = \int_{-\infty}^{+\infty} \frac{\nu(\sigma) \exp[-(z - \sigma)^2/4t']}{2(\pi t')^{\frac{1}{2}}} d\sigma, \tag{3.33}$$

$$\zeta(t) = \zeta_{\frac{1}{2}} t'^{\frac{1}{2}} + \zeta_1 t' + \zeta_{\frac{3}{2}} t'^{\frac{3}{2}} + \cdots, \tag{3.34}$$

where the $\zeta_{n/2}$ are constants, and

$$m(z) = \sum_{n=0}^{\infty} m_n z^n \tag{3.35}$$

and

$$\nu(z) = \sum_{n=0}^{\infty} \nu_n z^n \quad \text{for} \quad z < 0 \tag{3.36}$$

$$= \sum_{n=0}^{\infty} H_n z^n \quad \text{for} \quad z > 0. \tag{3.37}$$

Here, $v(z, \tau)$ is known for $z > 0$ so that the H_n are to be computed from (3.26) using (3.30).

Without going through all of the details, using conditions (3.17), (3.18), and (3.19), one finds

$$m_0 = 1, \tag{3.38}$$

$$m_1 \zeta_{\frac{1}{2}} = 0, \tag{3.39}$$

$$(2m_2/\beta)H_c = (\alpha/\beta)H_c + dw^*(0)/dt \qquad (t' = 0 \Rightarrow t = \tau) \tag{3.40}$$

$$m_1\zeta_1 + 2m_2/\beta = \alpha/\beta. \tag{3.41}$$

Similarly, using (3.33), (3.36), (3.37), and (3.13) yields

$$H_0 = (H_c - H_e)/H_c = \nu_0. \tag{3.42}$$

In order to obtain further information, as was mentioned earlier, (3.21) and (3.22) must also be used. This means computing derivatives of the expressions (3.32) and (3.33) and also the proper integrals involved. Again, by expanding for small time $(t - \tau)$, one finds

$$\zeta_{\frac{1}{2}} = 0, \tag{3.43}$$

$$\nu_1 = H_1, \tag{3.44}$$

$$\nu_2 = H_2, \tag{3.45}$$

$$\zeta_1 = -2H_2/H_1, \tag{3.46}$$

$$\nu_3 = \tfrac{1}{4}\pi^{\frac{1}{2}}\zeta_{\frac{3}{2}}H_1 + H_3 \tag{3.47}$$

by using (3.21) and (3.13) together. From (3.22)

$$m_1 = -\beta v_0[2h(\tau - l/v_0) - w(\tau)]. \tag{3.48}$$

Finally,

$$\zeta_{\frac{3}{2}} = \{-2v_0 h'(\tau - l/v_0) + (v_0 + \zeta_1)\zeta_1\beta v_0[2h(\tau - l/v_0) - w(\tau)] - 2H_2\zeta_1 - 6H_3\}(\tfrac{3}{4}\pi^{\frac{1}{2}}H_1)^{-1}. \tag{3.49}$$

Thus, ζ_1 and $\zeta_{\frac{3}{2}}$ are known in terms of H_1, H_2, H_3 and h and w. The H_n are by definition coefficients in the expansion of v about $z = 0$ for $t = \tau$, so that, using (3.26),

$$H_1 = -\frac{1}{\pi^{\frac{1}{2}}} \int_0^{\tau} \frac{1}{(\tau - \sigma)^{\frac{1}{2}}} \frac{\partial w}{\partial \sigma}(\sigma)\, d\sigma, \tag{3.50}$$

$$2H_2 = \partial w(\tau)/\partial t, \tag{3.51}$$

$$6H_3 = -\frac{1}{\pi^{\frac{1}{2}}} \int_0^{\tau} \frac{\partial^2 w}{\partial \sigma^2} \frac{1}{(\tau - \sigma)^{\frac{1}{2}}}\, d\sigma. \tag{3.52}$$

Observe that (3.50) and (3.29) are nearly the same and easily define H_1.

To gain an idea of what these formulas mean recall first of all that from (3.34) and the definition of real time, $T = (x_0^2/c^2)4\pi\sigma^N t = (x_0/c)v_0 t$, that

$$\xi(T + x_0 v_0 \tau/c) = x_0 \zeta(t + \tau) = x_0\left[\left(\frac{c}{x_0 v_0}\right)^{\frac{1}{2}} \zeta_{\frac{1}{2}} T^{\frac{1}{2}} + \frac{c}{x_0 v_0} \zeta_1 T + \left(\frac{c}{x_0 v_0}\right)^{\frac{3}{2}} \zeta_{\frac{3}{2}} T^{\frac{3}{2}} + \cdots\right]. \tag{3.53}$$

Further, consider a particular case for the driving signal, $h^*(t)$. Let $h^*(t)$ be the linear function

$$h^*(t) = H_c \delta t + H_e,$$

where $\delta < 0$. Then

$$h(t) = \delta t. \tag{3.54}$$

Then the expression for τ, the (dimensionless) time at which the critical field is first attained on $z = 0$, is most easily given in terms of the quantity

$$\bar{\tau} = v_0(\tau - l/v_0), \tag{3.55}$$

$$\bar{\tau}^2 - \frac{2}{\pi^{\frac{1}{2}}}\,\bar{\tau} + 1 - \frac{2}{\pi^{\frac{1}{2}}} \exp(\bar{\tau}^2) \int_{\bar{\tau}}^{\infty} \exp(-\lambda^2)\, d\lambda = \frac{v_0^2(H_e - H_c)}{2(-\delta)H_e} \equiv \mathcal{H}. \tag{3.56}$$

The quantities on the right are given, of course. In fact $(H_e - H_c)/(-\delta)H_e$ is nothing other than the time at which $h^*(t)$ becomes equal to H_c.

If

$$B(\bar{\tau}) = 1 - \frac{2}{\pi^{\frac{1}{2}}} \exp(\bar{\tau}^2) \int_{\bar{\tau}}^{\infty} \exp(-\lambda^2)\, d\lambda, \tag{3.57}$$

then

$$\zeta_1/v_0 = B(\bar{\tau})[2\bar{\tau}/\pi^{\frac{1}{2}} - B(\bar{\tau})]^{-1} \tag{3.58}$$

and

$$\frac{\zeta_{\frac{1}{2}}}{v_0^2} = \left[1 - B(\bar{\tau}) - \frac{\bar{\tau}}{v_0\pi^{\frac{1}{2}}} \int_0^1 \frac{f(\bar{\lambda})}{(1-\lambda)^{\frac{1}{2}}}\, d\lambda\right](\tfrac{3}{4}\pi^{\frac{1}{2}}[\bar{\tau}^2 - \mathcal{H}])^{-1}, \tag{3.59}$$

where

$$f(x) = (\pi x)^{-\frac{1}{2}} - v_0 \exp(v_0^2 x) \operatorname{erfc}(v_0 x^{\frac{1}{2}}), \tag{3.60}$$

$$\bar{\lambda} = (\bar{\tau}^2/v_0^2)\lambda,$$

and

$$\mathcal{H} = \tfrac{1}{2}v_0^2(H_e - H_c)/(-\delta)H_e. \tag{3.61}$$

The formulas (3.58) and (3.59) allow these coefficients to be computed for the special case of a linear input signal. The results of a sample calculation are recorded in Sec. 6 of this paper.

4. FORWARD SWITCHING BY A PROPAGATING MAGNETIC WAVE

In this section, the forward transition, from super to normal, will be calculated for the same external field behavior as in the backward switching problem of the previous section. This case is represented in Fig. 5. For $t < 0$, there will be a subcritical field, $H_0 < H_c$, imposed everywhere in the whole space. This means that for $z < 0$, the field will be H_0, and for $z > 0$, it will take on the half-space penetration distribution, $H_0 \exp(-\alpha^{\frac{1}{2}}z)$. Then, at $t = 0$, the field at $z = -l$ is raised so that it passes through H_c. This signal is propagated through the vacuum between $z = -l$ and $z = 0$, as described in Sec. 3 and carries the increasing field to the boundary of the superconducting region. When the critical value is reached on this boundary, at $t = \tau$, a normal front begins to move into the region $z > 0$. Again the whole problem may be broken up in three parts:

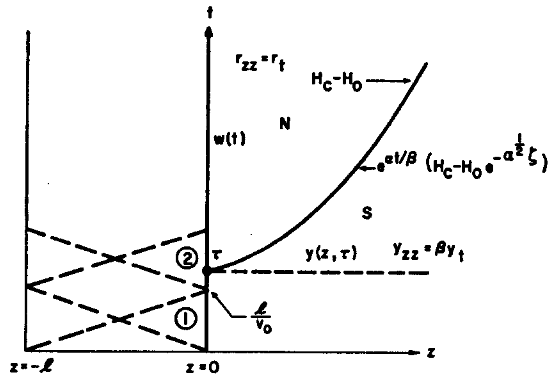

FIG. 5. Forward switching by a propagating plane magnetic wave (z–t space).

(1) For $z < 0$, the problem is quite similar to to that of 1° in Sec. 3. Letting $u = [H^0(z, t) - H_0]H_c^{-1}$, we have

$$u_{zz} = v_0^{-2}u_{tt}, \tag{4.1}$$

$$u(-l, t) = [h^*(t) - H_0]/H_c \equiv h(t) > 0, \tag{4.2}$$

$$u(z, 0) = u_t(z, 0) = 0, \tag{4.3}$$

$$u(0, t) = [w^*(t) - H_0]/H_c \equiv w(t). \tag{4.4}$$

Then in region 1,

$$u = w[(z + v_0t)/v_0], \tag{4.5}$$

and in region 2

$$u = w\left(\frac{z + v_0t}{v_0}\right) - h\left(\frac{z + v_0t - l}{v_0}\right) + h\left(\frac{-z + v_0t - l}{v_0}\right). \tag{4.6}$$

(2) In the superconducting region, if the magnetic field is $H^s(z, t)$ and we define

$$y(z, t) = \{[H^s(z, t) - H_0 \exp(-\alpha^{\frac{1}{2}}z)] \exp(\alpha\beta^{-1}t)\}H_c^{-1}, \tag{4.7}$$

the equations and boundary conditions become, for $0 \le t \le \tau$.

$$y_{zz} = \beta y_t, \tag{4.8}$$

$$y(z, 0) = 0, \tag{4.9}$$

$$y(0, t) = [w^*(t) - H_0]H_c^{-1} \exp(\alpha\beta^{-1}t) \equiv w(t) \exp(\alpha\beta^{-1}t) \equiv g(t), \tag{4.10}$$

$$w(0) = 0, \tag{4.11}$$

$$y(z, t) \to 0, \qquad z \to \infty. \tag{4.12}$$

Here $w^*(t)$ is the value of H^s at $z = 0$.

The solution for $z > 0$, $0 \leq t \leq \tau$, has the form

$$y(z, t) = \frac{z\beta^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \int_0^t \frac{g(\sigma) \exp[-z^2\beta/4(t-\sigma)]}{(t-\sigma)^{\frac{3}{2}}} d\sigma. \quad (4.13)$$

At the boundary, $z = 0$, for $t < \tau$, continuity of the electric field requires, for this case, that

$$V \int_0^t \frac{\partial u}{\partial z}(0^-, \gamma)\, d\gamma = \exp(-\alpha\beta^{-1}t) \frac{\partial y}{\partial z}(0^+, t) + \alpha \int_0^\infty \exp(-\alpha\beta^{-1}t) y(\gamma, t)\, d\gamma, \quad (4.14)$$

where $V = v_0 \times v_0'$. v_0 is as previously defined (2.3) and $v_0' = (4\pi x_0/c)\sigma^*$. Using the solution (4.13), we obtain

$$V \int_0^t \frac{\partial u}{\partial z}(0^-, \gamma)\, d\gamma = -\frac{\beta^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \exp(-\alpha\beta^{-1}t) \int_0^t \frac{\partial g}{\partial \sigma} \frac{1}{(t-\sigma)^{\frac{1}{2}}} d\sigma + \frac{\alpha}{\beta^{\frac{1}{2}}} \frac{\exp(-\alpha\beta^{-1}t)}{\pi^{\frac{1}{2}}} \int_0^t \frac{g(\sigma)}{(t-\sigma)^{\frac{1}{2}}} d\sigma. \quad (4.15)$$

Thus in region 1, Fig. 5, the left-hand side is essentially $w(t)$, the equation is homogeneous in w, and its unique solution, for $0 < t < l/v_0$, $z > 0$, is $w \equiv 0$, as one would expect.

However, in region 2, inserting $\partial u/\partial z$ yields

$$v_0'\left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} \exp(\alpha\beta^{-1}t)\left[w(t) - 2h\left(t - \frac{l}{v_0}\right)\right] = \int_0^t \frac{1}{(t-\sigma)^{\frac{1}{2}}}\left[\frac{\alpha}{\beta} g - \frac{\partial g}{\partial \sigma}\right] d\sigma. \quad (4.16)$$

Let

$$\exp(\alpha\beta^{-1}t) 2h(t - l/v_0) = \bar{f}(t). \quad (4.17)$$

Then

$$-v_0'\left(\frac{\pi}{\beta}\right)^{\frac{1}{2}}[g(t) - \bar{f}(t)] = \int_0^t \frac{1}{(t-\sigma)^{\frac{1}{2}}}\left[\frac{\partial g}{\partial \sigma} - \frac{\alpha}{\beta} g\right] d\sigma. \quad (4.18)$$

This integral equation may be solved by using Laplace transforms. The solution is in terms of a double convolution.

$$g(t) = \int_0^t \left\{\int_0^{t-s} \frac{v_0'}{\beta^{\frac{1}{2}}} \bar{f}(t - s - \sigma) \times \left[\delta(\sigma) - \frac{c}{(\pi\sigma)^{\frac{1}{2}}} + c^2 \exp(c^2\sigma) \operatorname{erfc}(c\sigma^{\frac{1}{2}})\right] d\sigma\right\} \times [(\pi s)^{-\frac{1}{2}} + d \exp(d^2 s) \operatorname{erfc}(-ds^{\frac{1}{2}})]\, ds, \quad (4.19)$$

where

$$\begin{cases} c = \frac{1}{2}\{[v_0'^2\beta^{-1} + 4\alpha/\beta]^{\frac{1}{2}} + v_0'\beta^{-\frac{1}{2}}\}, \\ d = \frac{1}{2}\{[v_0'^2\beta^{-1} + 4\alpha/\beta]^{\frac{1}{2}} - v_0'\beta^{-\frac{1}{2}}\}, \end{cases}$$

and $\delta(\sigma)$ is Dirac's delta function. At least formally this is the solution for $g(t)$, $l/v_0 < t > \tau$. At $t = \tau$, $g(\tau) = \exp(\alpha\beta^{-1}\tau)(H_c - H_0)$. Knowing $h(\tau)$, this gives an equation for τ.

(3) Again the stage is reached at which the free boundary begins. For simplicity, it is again assumed that $\tau < 2l/v_0$. Letting $r(z, t) = [H^N(z, t) - H_0]/H_c$ in the newly formed normal region, the boundary-value problem for r in the time interval $\tau < t < 2l/v_0$ is:

$$r_{zz} = r_t, \quad (4.20)$$

$$\begin{aligned} r(0, t) &= [w^*(t) - H_0]/H_c \equiv w(t), \\ w(\tau) &= [H_c - H_0]/H_c, \end{aligned} \quad (4.21)$$

$$r(\zeta(t), t) = (H_c - H_0)/H_c. \quad (4.22)$$

For $y(z, t)$ with $\tau < t < 2l/v_0$,

$$y_{zz} = \beta y_t, \quad (4.23)$$

$$y(z, t) = \frac{z\beta^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^t \frac{g(\sigma) \exp[-z^2\beta/4(t-\sigma)]}{(t-\sigma)^{\frac{3}{2}}} d\sigma, \quad (4.24)$$

$$H_c y(\zeta(t), t) = \exp(\alpha\beta^{-1}t)\{H_c - H_0 \exp[-\alpha^{\frac{1}{2}}\zeta(t)]\}, \quad (4.25)$$

where $\zeta(t) \equiv 0$ for $t < \tau$. It will be useful again to let $t' = t - \tau$, $\zeta = \zeta(t')$, and then for $t' = 0$, $\zeta(0) = 0$. The same calculational procedure will be adopted as was used in Sec. 3: $r(z, t)$ and $y(z, t)$ are defined by means of heat equation singularity distributions

$$r(z, t) = \int_{-\infty}^{+\infty} \frac{R(\sigma) \exp[-(z-\sigma)^2/4t']}{2(\pi t')^{\frac{1}{2}}} d\sigma, \quad (4.26)$$

$$y(z, t) = \beta^{\frac{1}{2}} \int_{-\infty}^0 \frac{Y(\sigma) \exp[-(z-\sigma)^2\beta/4t']}{2(\pi t')^{\frac{1}{2}}} + \beta^{\frac{1}{2}} \int_0^\infty \frac{\mathcal{Y}(\sigma) \exp[-(z-\sigma)^2\beta/4t']}{2(\pi t')^{\frac{1}{2}}} d\sigma. \quad (4.27)$$

Again $y(z, \tau)$ is known for $z > 0$ so that $\mathcal{Y}(\sigma)$ is known [in terms of (4.13) and (4.16)]. The conditions on y and r given above [(4.21), (4.22), (4.25)] are not sufficient to determine these functions and the free boundary ζ. The continuity of $E(z, t)$ at $z = \zeta(t')$ and $z = 0$ must also be imposed. This gives two additional relations:

At $z = \zeta$

$$\beta \frac{\partial r}{\partial z}(\zeta(t'), t) = \exp(-\alpha\beta^{-1}t)\frac{\partial y}{\partial z}(\zeta(t'), t)$$
$$+ \alpha \int_{\zeta(t')}^{\infty} \exp(-\alpha\beta^{-1}t)y(\sigma, t)\, d\sigma. \tag{4.28}$$

At $z = 0$,

$$-\partial r(0, t)/\partial z = v_0[2h(t - l/v_0) - w(t)]. \tag{4.29}$$

Now the functions to be found will be expanded in power series and a solution sought for t' small.

$$R(\sigma) = R_0 + R_1\sigma + R_2\sigma^2 + \cdots, \tag{4.30}$$
$$Y(\sigma) = Y_0 + Y_1\sigma + Y_2\sigma^2 + \cdots, \tag{4.31}$$
$$\zeta(t') = \zeta_{\frac{1}{2}}t'^{\frac{1}{2}} + \zeta_1 t' + \zeta_{\frac{3}{2}}t'^{\frac{3}{2}} + \cdots. \tag{4.32}$$

Also the known function $\mathcal{Y}(\sigma)$ is expanded

$$\mathcal{Y}(\sigma) = \mathcal{Y}_0 + \mathcal{Y}_1\sigma + \mathcal{Y}_2\sigma^2 + \cdots, \tag{4.33}$$
$$\mathcal{Y}_0 = (H_c - H_0)\exp(\alpha\beta^{-1}\tau)/H_c. \tag{4.34}$$

These expansions are inserted into (4.26) and (4.27), and condition (4.21) is found to be satisfied with

$$R_0 = (H_c - H_0)/H_c, \tag{4.35}$$
$$R_2 = \tfrac{1}{2}(\partial w/\partial t')|_{t'=0}. \tag{4.36}$$

Imposing (4.22) requires

$$R_1\zeta_{\frac{1}{2}} = 0, \tag{4.37}$$
$$R_1\zeta_1 + \frac{R_2}{\pi^{\frac{1}{2}}}\int_{-\infty}^{+\infty}(\zeta - 2\alpha)^2\exp(-\alpha^2)\,d\alpha = 0, \tag{4.38}$$
$$R_1\zeta_{\frac{3}{2}} + \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{+\infty}[2R_2\zeta_1(\zeta_{\frac{1}{2}} - 2\alpha)$$
$$+ R_3(\zeta_{\frac{1}{2}} - 2\alpha)^3]\exp(-\alpha^2)\,d\alpha = 0. \tag{4.39}$$

The continuity condition at $z = 0$, (4.29) requires

$$R_1 = -v_0[2h(\tau - l/v_0) - w(\tau)], \tag{4.40}$$
$$6R_3 = -v_0[2h'(\tau - l/v_0) - w'(\tau)]. \tag{4.41}$$

(Primes refer to forward derivatives with respect to the argument.) Thus, in (4.37), $\zeta_{\frac{1}{2}} = 0$, and (4.38) is simplified. In a similar manner, using (4.22), (4.25), and (4.28), one obtains

$$Y_0 = \mathcal{Y}_0 = \exp(\alpha\beta^{-1}\tau)(H_c - H_0)/H_c, \tag{4.42}$$
$$Y_1 = \mathcal{Y}_1, \qquad Y_2 = \mathcal{Y}_2, \tag{4.43}$$
$$\zeta_1 = \frac{-2\mathcal{Y}_2 - \exp(\alpha\beta^{-1}\tau)\cdot\alpha(H_c - H_0)/\beta H_c}{\mathcal{Y}_1 - \alpha^{\frac{1}{2}}(H_0/H_c)\exp(\alpha\beta^{-1}\tau)}, \tag{4.44}$$

and finally,

$$\zeta_{\frac{3}{2}} = \frac{4(Y_3 - \mathcal{Y}_3)}{\pi^{\frac{1}{2}}\beta^{\frac{3}{2}}(\mathcal{Y}_1 - \alpha^{\frac{1}{2}}\exp(\alpha\beta^{-1}\tau)H_0/H_c)}. \tag{4.45}$$

The final expressions for ζ_1 and $\zeta_{\frac{3}{2}}$ are similar to those obtained for the backward switching in Sec. 3. For the computation of ζ_1

$$\mathcal{Y}_1 = \left.\frac{\partial y}{\partial z}\right|_{z=0} = -\left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}\frac{1}{(\tau - \sigma)^{\frac{1}{2}}}\frac{\partial g}{\partial \sigma}\,d\sigma, \tag{4.46}$$
$$2\mathcal{Y}_2 = \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=0} = \beta\left.\frac{\partial g}{\partial \sigma}\right|_{\sigma=\tau}, \tag{4.47}$$

where g is given by (4.19). Recall, however, that g is obtained from the integral equation (4.18). From the integral equation,

$$-\left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}\frac{1}{(\tau - \sigma)^{\frac{1}{2}}}\frac{\partial g}{\partial \sigma}\,d\sigma = \frac{\beta}{v_0'}\left[\frac{\partial g}{\partial t}(\tau) - \frac{\alpha}{\beta}g(\tau)\right]$$
$$- \left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}\frac{1}{(\tau - \sigma)^{\frac{1}{2}}}\bar{f}(\sigma)\,d\sigma. \tag{4.48}$$

Thus

$$\mathcal{Y}_1 = \frac{\beta}{v_0'}\left[\frac{\partial g}{\partial t}(\tau) - \frac{\alpha}{\beta}g(\tau)\right]$$
$$- \left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}\frac{1}{(\tau - \sigma)^{\frac{1}{2}}}\left[\exp(\alpha\beta^{-1}\sigma\cdot 2h\left(\sigma - \frac{l}{v_0}\right)\right]'\,d\sigma. \tag{4.49}$$

(The prime over the bracket in the last integral means differentiation with respect to σ.)

The formulas given above have been employed in computing the value of ζ_1 for the case of a linear input signal. These may then be compared with similar values obtained for the "backward" process of Sec. 3. The results are described in Sec. 6.

5. BACKWARD SWITCHING BY A DISCONTINUOUS MAGNETIC FIELD

In this section another backward switching problem is considered. Here, however, the external region, $z < 0$, is taken to be a good conductor. The governing equation for the magnetic field will then be

$$H^0_{zz} = \beta^0 H^0_t, \qquad (\beta^0 = \sigma^0/\sigma^N). \tag{2.3a}$$

It is assumed, in deriving (2.3a) that σ^0 is comparable to σ^N and that wave effects may be neglected. The particular external magnetic driving condition that is considered is one in which the field at $t = 0$ is instantaneously changed from H_a to $H_a < H_c < H_a$ for all $z < 0$ and to $H_0 \leq H_c$ at $z = 0$ (Fig. 6). A singularity is thus imposed at $t = 0$, $z = 0$, and one may expect the boundary motion to begin in in a singular fashion. This starting condition is imposed, however, in analogy to the problems which have been solved for the "forward" transition in which it is only necessary to specify the constant value of H on $z = 0$.[5,6]

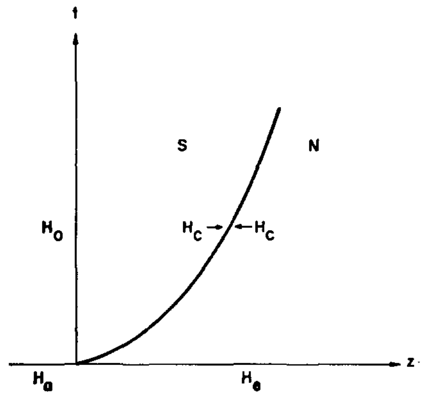

FIG. 6. Backward switching by a discontinuous external magnetic field (z–t space).

Equation (2.4) for the conducting external region may be rewritten as

$$\beta \frac{\partial H^N}{\partial z}(\zeta, t) - \frac{\partial H^S}{\partial z}(\zeta, t) = \frac{\beta}{\beta^0}\frac{\partial H^0}{\partial z}(0, t) - \frac{\partial H^S}{\partial z}(0, t) - \alpha \int_0^{\zeta(t)} H^S(z, t)\, dz. \tag{5.1}$$

The other conditions are :

$$H^S(0, t) = H_0, \tag{5.2}$$

$$H^S(\zeta(t), t) = H_c = H^N(\zeta(t), t), \tag{5.3}$$

$$H^N(z, 0) = H_e. \tag{5.4}$$

The solution for the external region may be given without difficulty since it is of standard heat equation form:

$$H^0(z, t) = H_a - \frac{z(\beta^0)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \times \int_0^t (H_0 - H_a)\frac{\exp\left[-z^2\beta^0/4(t-\sigma)\right]}{(t-\sigma)^{\frac{3}{2}}}\, d\sigma. \tag{5.5}$$

From this

$$\lim_{z\to 0^-} \frac{\partial H^0}{\partial z} = \frac{1}{\pi^{\frac{1}{2}}}\frac{(\beta^0)^{\frac{1}{2}}}{t^{\frac{1}{2}}}\left[H_0 - H_a\right]. \tag{5.6}$$

Setting $H^S(z, t) = \exp(-\alpha\beta^{-1}t)v(z, t)$ and introducing this and (5.6) into (5.1), we get

$$\beta\frac{\partial H^N}{\partial z}(\zeta, t) - \exp(-\alpha\beta^{-1}t)\frac{\partial v}{\partial z}(\zeta, t) = \frac{\beta}{(\beta^0)^{\frac{1}{2}}}\frac{1}{\pi^{\frac{1}{2}}}\frac{1}{t^{\frac{1}{2}}}(H_0 - H_a) - \exp(-\alpha\beta^{-1}t)\frac{\partial v}{\partial z}(0, t) - \alpha\int_0^{\zeta(t)} \exp(-\alpha\beta^{-1}t)v(z, t)\, dz. \tag{5.7}$$

Note that v must now satisfy

$$v_{zz} = \beta v_t \tag{5.8}$$

with $v(0, t) = \exp(\alpha\beta^{-1}t)H_0$, $v(\zeta(t), t) = \exp(\alpha\beta^{-1}t)H_c$, while H^N satisfies

$$H^N_{zz} = H_t \tag{5.8a}$$

with $H(0, t) = H_e$, $H(\zeta(t), t) = H_c$.

At this point it is again necessary to turn to approximations. In this case, the small-time method is used to obtain the initial behavior. It is also possible to obtain a large-time approximation.

A. The Small-Time Approximation

For very small times, since $\exp(-\alpha\beta^{-1}t) \sim 1$ and $\int_0^{\zeta(t)} \exp(-\alpha\beta^{-1}t)v\, dz \sim o(1)$, it appears that a Stefan-like similarity condition will prevail. This is indicated by the presence of the $t^{-\frac{1}{2}}$ term on the right-hand side of (5.7). If $\zeta(t)$ is, as before, assumed to have the form

$$\zeta(t) = \zeta_{\frac{1}{2}}t^{\frac{1}{2}} + \zeta_1 t + \zeta_{\frac{3}{2}}t^{\frac{3}{2}} + \cdots, \tag{5.9}$$

where the $\zeta_{i/2}$ are constants. Then H^N and v will be taken as

$$H^+(z, t) = H^+(zt^{-\frac{1}{2}}), \qquad v = v(zt^{-\frac{1}{2}}). \tag{5.10}$$

All of these representations are for $t \ll 1$. Let $zt^{-\frac{1}{2}} = s$, then take

$$H^+ = A_1\int_s^\infty \exp(-\tfrac{1}{4}w^2)\, dw + B_1 \tag{5.11}$$

and

$$v = A_2\int_0^{s\beta^{\frac{1}{2}}} \exp\left(\frac{-w^2}{4}\right) dw + B_2, \tag{5.12}$$

where A_1, A_2, B_1, B_2 are constants. If, to these solutions the boundary and initial conditions are applied, it is found that the time-dependent boundary conditions cannot be exactly satisfied for constant values of A_1, A_2, and B_2. In this case the choice is made to allow A_1, A_2, and B_2 to take on the time-dependent values which do satisfy the boundary conditions. This, of course, means that the H^+ and v of (5.11) and (5.12) are no longer exact solutions of (5.8) and (5.8a), but they will be approximate solutions for very small time.[13] Bearing this in mind, we have

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[13] In Secs. 3, 4 and in Ref. 4 exact solutions to the equations corresponding to (5.8) and (5.8a) were chosen and the boundary conditions were only approximately satisfied. It can be shown that the two approximations are equivalent for very small times.

$$B_1 = H_e, \qquad A_1 = (H_c - H_e)\left[\int_{\zeta/t^{\frac{1}{2}}}^{\infty} \exp\left(\frac{-w^2}{4}\right) dw\right]^{-1} \tag{5.13}$$

and

$$B_2 = H_0 \exp(\alpha\beta^{-1}t), \qquad A_2 = (H_c - H_0)\exp(\alpha\beta^{-1}t)\left[\int_0^{\zeta\beta^{\frac{1}{2}}/t^{\frac{1}{2}}} \exp\left(\frac{-w^2}{4}\right) dw\right]. \tag{5.13a}$$

H^+ and v in this form are now substituted into (5.7) and the terms expanded in powers of $t^{\frac{1}{2}}$. Without going into details, the results are as follows:

Using terms of order $t^{-\frac{1}{2}}$,

$$\frac{(H_e - H_c)\exp[-(\frac{1}{2}\zeta_{\frac{1}{2}})^2]}{\int_{\zeta_{\frac{1}{2}}/2}^{\infty} \exp(-w^2)\, dw} + \frac{(H_c - H_0)(1 - \exp[-(\frac{1}{2}\zeta_{\frac{1}{2}})^2\beta])}{\beta^{\frac{1}{2}}\int_0^{\frac{1}{2}\zeta_{\frac{1}{2}}\beta^{\frac{1}{2}}} \exp(-w^2)\, dw} - \frac{2}{\pi^{\frac{1}{2}}}\frac{(H_0 - H_a)}{(\beta^0)^{\frac{1}{2}}} = 0 \tag{5.14}$$

This is a transcendental equation for $\zeta_{\frac{1}{2}}$ which may be solved numerically.

It is easy to show that for $\zeta_{\frac{1}{2}}$ to be positive, the condition

$$(\beta^0)^{\frac{1}{2}}(H_e - H_c)/(H_0 - H_a) < 1$$

must prevail. If this condition does hold, then there will exist a unique solution for $\zeta_{\frac{1}{2}}$. Furthermore, for a given value of $H_e - H_c$, the largest value of $\zeta_{\frac{1}{2}}$ will occur for $H_c = H_0$. It is possible to choose H_e, H_c, H_0, and H_a so that $\zeta_{\frac{1}{2}}$ takes on any value between zero and infinity.

The next set of terms, of order t^0, yields a relation for ζ_1

$$\begin{aligned}&(H_c - H_e)\frac{\exp[-(\frac{1}{2}\zeta_{\frac{1}{2}})^2]}{D_1(0)}\left\{a + \frac{D_1'(0)}{D_1(0)}\right\}\\ &\quad + (H_c - H_0)\frac{\exp[-(\frac{1}{2}\zeta_{\frac{1}{2}})^2\beta]}{\beta^{\frac{1}{2}}D_2(0)}\left\{a + \frac{D_2'(0)}{D_2(0)}\right\}\\ &= \frac{(H_c - H_0)}{\beta^{\frac{1}{2}}}\frac{D_2'(0)}{[D_2(0)]^2},\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}D_1(0) &= 2\int_{\frac{1}{2}\zeta_{\frac{1}{2}}}^{\infty} \exp(-\alpha^2)\, d\alpha,\\ D_1'(0) &= -\zeta_1 \exp[-\tfrac{1}{4}(\zeta_{\frac{1}{2}})^2],\\ D_2(0) &= 2\int_0^{\beta^{\frac{1}{2}}(\zeta_{\frac{1}{2}}/2)} \exp(-\alpha^2)\, d\alpha,\\ D_2'(0) &= \zeta_1\beta^{\frac{1}{2}}\exp[-\tfrac{1}{4}\beta(\zeta_{\frac{1}{2}})^2],\\ a &= \tfrac{1}{2}\zeta_{\frac{1}{2}}\zeta_1.\end{aligned} \tag{5.16}$$

Note that if $H_c = H_0$, then unless $\zeta_1 = 0$, there will be a relation for $\zeta_{\frac{1}{2}}$ which is incompatible with (5.14). Thus, for the case $H_c = H_0$, there will be no term of order t. The process may now be continued to obtain higher-order terms.

Equations (5.14) and (5.15) may be solved numerically for $\zeta_{\frac{1}{2}}$ and ζ when $(H_e - H_c)$, $(H_c - H_0)$, $(H_0 - H_a)$, β, and β^0 are given. This has been carried out and is described along with the other cases in Sec. 6.

B. Large-Time Expansion

The computation set out above holds for small values of time. In what follows an approximation is obtained for large values of αt. Again this discussion is very similar to the corresponding one in Ref. 4 so that details will not be given. The process will be to find a solution for large αt for $H^S(z, t)$ which satisfies to certain orders of approximation the boundary conditions and the differential equation. This solution is then inserted into (5.7) and a boundary-value problem for $H^N(z, t)$ is obtained. Again H^0 is taken to be the function given in (5.5).

The large-time approximation is made with the idea in mind that when the superconducting region has spread somewhat into the half-space, the magnetic field value must still accommodate the boundary values, $H^s(0, t) = H_0$ and $H^s(\zeta(t), t) = H_c$. On the other hand, in the middle of its region, the value of $H^s(z, t)$ should be quite low. In fact, for very large values of αt, one may expect the magnetic field to be zero except near the boundaries. A solution will be sought, therefore, that has these properties. It is convenient to set $t = \beta t^*$. Then (2.2a) becomes

$$H^-_{zz} = \alpha H^- + H_{t^*}. \tag{5.17}$$

Now let

$$\begin{aligned}H^-(z, t^*) &= a(z, t^*)\exp[\alpha^{\frac{1}{2}}\phi(z, t^*)]\\ &\quad + b(z, t^*)\exp[\alpha^{\frac{1}{2}}\psi(z, t^*)],\end{aligned} \tag{5.18}$$

and expand a, b, and ζ in negative half powers of α:

$$a(z, t^*) = a_0(z, t^*) + \frac{a_1(z, t^*)}{\alpha^{\frac{1}{2}}} + \frac{a_2(z, t^*)}{\alpha}, \tag{5.19}$$

$$b(z, t^*) = b_0(z, t^*) + \frac{b_1(z, t^*)}{\alpha^{\frac{1}{2}}} + \frac{b_2(z, t^*)}{\alpha} + \cdots, \tag{5.20}$$

$$\zeta(t^*) = w_0(t^*) + \frac{w_{\frac{1}{2}}(t^*)}{\alpha^{\frac{1}{2}}} + \frac{w_1(t^*)}{\alpha} + \cdots. \tag{5.21}$$

These expressions are substituted into (5.17) and into the boundary conditions that apply to H^s in (5.12) and (5.13). Like powers of α are equated with the following results:

$$\phi = -z, \tag{5.22}$$

$$\psi = z - \zeta(t^*), \tag{5.23}$$

and

$$H^-(z, t^*) = H_0 \exp(-\alpha^{\frac{1}{2}} z) + H_c \exp\{\gamma(z, t^*)[1 + \omega(z, t^*)]\}, \tag{5.24}$$

where

$$\gamma(z, t^*) = \tfrac{1}{2} w_0(t^*)\,(w_0(t^*) - z) + \alpha^{\frac{1}{2}}(z - \zeta(t^*)), \tag{5.25}$$

$$\omega(z, t^*) = \tfrac{1}{2}\alpha^{-\frac{1}{2}}[(w_0 - z)(\dot{w}_1 - \tfrac{1}{4}\dot{w}_0^2) - \tfrac{1}{4}\ddot{w}_0(z - w_0)^2] + O(\alpha^{-1}) + \cdots. \tag{5.26}$$

Dots refer to differentiation with respect to t^*.

Inserting this into (5.7), we get

$$\frac{\partial H^+}{\partial z}(\zeta, t) = -H_c\dot{\zeta}(t) + \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} (\beta^0)^{-\frac{1}{2}}(H_0 - H_a) + O(e^{-\alpha^{\frac{1}{2}}\zeta}). \tag{5.27}$$

Observe that for $H_0 = H_a$, there can be no forward motion of the boundary, as has been mentioned before.

The boundary value problem for H^N [Eqs. (5.8a) and (5.27)] is now a Stefan problem, and the solution can be written down as

$$H^+(z, t) = (H_c - H_e)\left[\int_{\zeta t^{-\frac{1}{2}}}^{\infty} \exp(-\tfrac{1}{4}\alpha^2)\, d\alpha\right]^{-1} \times \int_{zt^{-\frac{1}{2}}}^{\infty} \exp(-\tfrac{1}{4}\alpha^2)\, d\alpha + H_e. \tag{5.28}$$

If $\zeta(t) = w_0 t^{\frac{1}{2}}$, then this implies that

$$\exp(-\tfrac{1}{4}w_0^2)(H_e - H_c)\left[2\int_{\frac{1}{2}w_0}^{\infty} \exp(-\alpha^2)\, d\alpha\right]^{-1} = -\tfrac{1}{2}w_0 H_c + (\pi\beta^0)^{-\frac{1}{2}}(H_0 - H_a). \tag{5.29}$$

We observe here that, in order for w_0 to be positive, the condition

$$(\beta^0)^{\frac{1}{2}}(H_e - H_c)/H_0 - H_a = C < 1$$

must again prevail. If this condition holds, then there will exist a unique solution for Eq. (5.29), which, however, never exceeds the quantity

$$(2/H_c\pi^{\frac{1}{2}})[(\beta^0)^{\frac{1}{2}}(H_0 - H_a) - (H_e - H_c)].$$

Given β^0, $(H_e - H_c)$, $(H_0 - H_a)$ (such that $C < 1$), Eq. (5.29) may be solved numerically. This has been carried out and is described with other results in Sec. 6.

6. NUMERICAL RESULTS

In this section numerical results for the three cases discussed in the preceding sections are given. Only first terms in the expansions have been produced.

For the "backward" switching—normal to super—computations can be made directly from formula (3.58). Here, as in the forward switching case, only the linear input signal (3.54) has been considered. For forward switching it was found that the use of (4.19) in (4.44) was too complicated. Approximations were employed in solving the integral equation (4.18), which seemed suitable for the degree of accuracy needed in the comparison to be carried out here.

In particular, recalling (4.10) that $g(t) = w(t)\exp(\alpha\beta^{-1}t)$, then the integral equation (4.18) may be rewritten as

$$w(t) - 2h\left(t - \frac{l}{v_0}\right) = -\frac{1}{v_0\pi^{\frac{1}{2}}}\int_0^t \frac{\exp[-\alpha(t-\sigma)]}{(t-\sigma)^{\frac{1}{2}}}\frac{\partial w}{\partial\sigma}(\sigma)\, d\sigma, \tag{6.1}$$

where now, specifically, $\beta = 1$ and $v_0' = v_0$. Approximate solutions for α large have the form

$$w(\tau) = 2\delta\left(\tau - \frac{l}{v_0}\right)\left[1 - \left(\frac{1}{2v_0 l}\right)^{\frac{1}{2}}\right] \text{ if } \frac{v_0}{l} \gg \alpha, \tag{6.2}$$

$$w(\tau) = 2\delta\left(\tau - \frac{l}{v_0}\right)\left[1 - \left(\frac{1}{2v_0 l}\right)^{\frac{1}{2}} \times \left[2\int_0^{\alpha(\tau - l/v_0)^{\frac{1}{2}}} \exp(-\eta^2)\, d\eta\right]^{-1}\right] \text{ if } v_0/l \sim \alpha. \tag{6.3}$$

Using these, $\mathcal{Y}_2$ and $\mathcal{Y}_1$ may be approximated for use in (4.44) and ζ_1 calculated.

Instead of giving both $H_e - H_c$ and δ for the backward case or $H_c - H_0$ and δ for the forward it was easier to choose values of l. For all cases τ was arbitrarily taken to be $\tau = \frac{3}{2}l/v_0$ (see Figs. 4 and 5). v_0 has a numerical value equal to 5×10^9 cm/sec for the cases in mind ($\sigma^N = 10^{19}$ sec^{-1}, $x_0 = 1$ cm, $c = 3 \times 10^{10}$ cm/sec). In addition, H_0 and H_e were chosen so that

$$H_0 = \tfrac{1}{3}H_c, \qquad H_e = 3H_c. \tag{6.4}$$

Then the calculations may be summarized in Table I.

It is clear that in the backward case (normal to super), as the rise time of the input signal, $1/\delta$, increases, the initial speed of the transition decreases according to the relation $\zeta_1 \sim \delta^{\frac{1}{2}}$. On the other hand, it can be shown that as δ decreases in the forward case (super to normal), ζ_1 approaches $\frac{4}{3}\alpha^{\frac{1}{2}}$ as a limiting value.

A very definite difference between the starting speeds is thus brought out. Although this result applies only to the transient state and then only to its initial portion, it is probably also true that a steady state signal will meet with this kind of nonlinear behavior. In fact, if the incoming signal from the distance l were a sawtooth wave, the present analysis implies that the downward stroke of the wave would produce a slow-moving backward-phase boundary, whereas the upward portion would produce a faster one. This would hold except for the case where the slope is extremely steep, and here the transitions would move at about the same speed.

In the case of backward switching when the external region is a good but nonsuperconducting conductor, the pertinent variables are

$$p = (\beta^0)^{\frac{1}{2}}(H_e - H_c)/(H_0 - H_a),$$

$$q = (\beta^0)^{\frac{1}{2}}(H_e - H_0)/(H_0 - H_a).$$

Then, if one sets $E(x) = 2/\pi^{\frac{1}{2}} \int_0^x \exp(-\alpha^2)d\alpha$, with $\zeta_{\frac{1}{2}} = 2x$, the equation for x becomes

$$pe^{-x^2}/[1 - E(x)] + q(1 - e^{-x^2})/E(x) = 1.$$

For $q = 0$:

p	x
1.0	0
0.75	0.27
0.50	0.75
0.25	2.07.

For $q = 1$:

p	x
1.0	0
0.75	0.145
0.50	0.345
0.25	0.65.

The values of x or $\zeta_{\frac{1}{2}}$ can be interpreted, of course, as a set of parabolas in the x, t plane, which will be modified by higher-order terms, as they are in Ref. 5.

In the asymptotic calculation for this case, the parameters are p again, as defined above, and

$$Q = H_c\pi^{\frac{1}{2}}(\beta^0)^{\frac{1}{2}}/(H_0 - H_a).$$

For the special case $Q = 2\pi^{\frac{1}{2}}$ the following values are obtained:

p	x
0.75	0.057
0.50	0.121
0.25	0.196.

If one choses the particular values

$$\beta^0 = 1,\ H_0 - H_a = 1,\ H_e - H_0 = 1,\ H_c = 2,$$

then $q = 1$, $Q = 2\pi^{\frac{1}{2}}$. For the same value of p, the values of x in the asymptotic case are much lower than in the initial portion. (Compare: $q = 1$, $p = 0.75$, $x = 0.145$ with $Q = 2\pi^{\frac{1}{2}}$, $p = 0.75$, $x = 0.57$.) This indicates the slowing down of the transition, which is to be expected as the superconducting region grows larger.

TABLE I. Summary of calculating of ζ_1.[a]

		Backward	Forward
l	δ	ζ_1	ζ_1
1	10^{10}	$\pi^{\frac{1}{2}}/2 \times 10^5$	$\pi^{\frac{1}{2}}/2 \times 10^5$
100	10^8	$\pi^{\frac{1}{2}}/2 \times 10^4$	2.1×10^4
1000	10^7	$3\pi^{\frac{1}{2}}/2 \times 10^3$	1.4×10^4

[a] $\alpha = 10^8$; $\tau = 3l/2v_0$; $v_0 = 5 \times 10^9$; $\xi_1 = c\zeta_1/v_0 = 6\zeta_1$ cm/sec.

Structure of Fermion Density Matrices. II. Antisymmetrized Geminal Powers

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Functions which can be expressed as an antisymmetrized power of a single two-particle function (a *geminal*) occur in the BCS ansatz. They constitute a comparatively tractable generalization of a single Slater determinant. The second-order reduced density matrix and bounds on its eigenvalues are obtained for the most general such antisymmetrized geminal power (AGP). These eigenvalues are physically of great importance, being analogs, in an analysis of the wavefunction into pair states, of the one-particle occupation numbers of the independent-particle model. A necessary and sufficient condition is given that an alleged first-order reduced density matrix be derivable from an N-particle AGP function. It is shown that only for exceptional functions (called *extreme*) is the generating geminal of the AGP function an eigenfunction of the 2-matrix. The extreme 2-matrix is diagonalized explicitly and provides a vivid example of the fact that one-particle occupation numbers *alone* are unable to convey certain information of decisive significance about the wavefunction. The extreme 2-matrix satisfies the assumption basic to current theories of superconductivity.

I. INTRODUCTION

THE most formidable obstacle to the application of quantum mechanics to systems of more than two fermions is the difficulty of describing the wavefunction. If the description is attempted numerically then this difficulty increases exponentially with N, the number of particles. This purely mathematical difficulty is one of the chief justifications for the widespread appeal to the independent particle model and the Hartree–Fock approximation. As is well-known, the overlap between a single Slater determinant and the wavefunction of any system with realistic interactions decreases exponentially with N. Already in 1935, the inadequacy of the independent particle picture was apparent to Eddington whose "added particle" discussed in Sec. 14.2 of his book[1] and later called the "top particle" is a precursor of the current method of quasiparticles.

If $g(12)$ is an antisymmetric two-particle function of rank N (i.e., expressible by means of N one-particle functions) then the antisymmetrized product $A_N[g(12)g(34) \cdots g(N-1, N)]$ is an N-particle function of rank N and, therefore, a single Slater determinant. If the rank of the geminal (a two-particle function, as an *orbital* is a one-particle function) exceeds N, then the above function is more general than a single Slater determinant and appropriate for the analysis of a wavefunction into geminals when a particular geminal is more important than any other. Blatt[2] has argued, quite cogently, that the assumption that the ground state of a superconductor can be approximated by such an antisymmetrized geminal power (AGP) is the key hypothesis of the BCS–Bogoliubov–Quasi Chemical Equilibrium theories, being even more important than the approximate form assumed for the Hamiltonian.

Functions of AGP type also occurred in the paper,[3] referred to as SFDM-I, where it was proved that only for *extreme* AGP functions does the largest eigenvalue, λ_1^2, of the 2-matrix (second-order reduced density matrix) approach its least upper bound. Penrose,[4] Penrose and Onsager,[5] and Yang[6] have shown that "large" eigenvalues are associated, in ways still not fully understood, with a variety of condensation and long-range-order phenomena in superfluids and superconductors. Particularly noteworthy is Yang's argument that the assumption of off-diagonal-long-range-order (ODLRO) in the 2-matrix explains flux quantization in superconducting rings and that ODLRO is accompanied by a large value of λ_1^2.

In the present paper, we study the 1-matrix and the 2-matrix of AGP functions of an even number of particles, N. It should not be difficult to extend our results to the case of N odd by the method of Sec. 7 of SFDM-I. Theorem 2.1 gives the AGP 2-matrix explicitly, and Theorem 2.2 provides new bounds on its eigenvalues. As announced in Sec. 7 of SFDM-I, Theorem 2.1 states that the natural orbitals of the AGP function g^N are the same as those of g. Theorem 3.2 gives a necessary and suf-

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ficient condition that an alleged 1-matrix corresponds to an AGP function. In Sec. IV, *extreme* functions are characterized among all AGP functions by a rather surprising property which throws some light on the reason that geminals are so much harder to work with than orbitals.

We note also that Eq. (4.10) below, which seems to be a key assumption of current microscopic theories of superconductivity, is *exactly* valid for extreme AGP functions.

Theorem 4.2 achieves the diagonalization of the 2-matrix of an extreme AGP function and illustrates the fact that, contrary to our experience with wavefunctions of Slater type, there is essential information about a wavefunction which is not conveyed by the orbital occupation numbers alone. Thus the statement "To define an eigenstate of the whole system, it is sufficient to indicate which plane waves are occupied by means of a distribution function $n(k)$, " which occurs on p. 1 of Nozières' excellent book[7] is incorrect, in general. It is correct *only* for a single Slater determinant but the wavefunction of no known physical system is described correctly by a single Slater determinant. This misconception, which is a carryover of the inadequate description of atomic energy levels as single configurations, seems rather widespread among the devotees of second quantization and issues in unnecessary confusion verging on mysticism.

A subsequent paper, SFDM-III, on the statistics and energy of interacting fermions will enlarge on a recent note[8] which announced a new form of Yang's criterion for the onset of super properties. In SFDM-IV, we shall prove the theorem, announced in part at the Sanibel Island Conference and, completely, at the Institute for Advanced Studies in January 1964, giving necessary and sufficient conditions for the N-representability of the 2-matrix.

In order to make this paper quasiindependent of SFDM-I, we review and refine the most important definitions. For an N-fermion system in a pure state Ψ, the *density operator*, D^N, or *N-operator* is an integral operator whose kernel is the *N-matrix*, $D^N(x;x') = \Psi(x)\Psi^*(x')$ where x stands for the space and spin variables of all N particles; the *2-operator* is an integral operator, $D^2 = D^2(\Psi)$ whose kernel is the *2-matrix* (two-particle reduced density matrix), $D^2(12;1'2') = D^2(x_1x_2;x_1'x_2') = \int \Psi(x_1x_2\cdots x_N)\Psi^*(x_1'x_2'x_3\cdots x_N)d(x_3\cdots x_N)$; the *1-matrix*, $D^1(1;1') = \int D^2(x_1x_2;x_1'x_2)\,dx_2$; the *p-matrix* is defined analogously; the natural p-states of Ψ are the eigenfunctions of D^p, are denoted by α_i^p and have corresponding eigenvalues λ_i^p; g is a *geminal*, that is a two-particle function; g^N is a *normalized* AGP function of N fermions obtained by antisymmetrizing $g(12)g(34)\cdots g(N-1,N)$. A one-particle function is called an *orbital*, so α_i^1 are *natural orbitals* or *norbs.* The reader should distinguish carefully between λ_σ, λ_σ^1, and λ_σ^2 which are, respectively, eigenvalues of $D^1(g)$, $D^1(g^N)$, and $D^2(g^N)$. The number of nonzero eigenvalues of $D^1(g)$, counting their multiplicities is the *rank*, r, of g and of g^N. When r is finite it is even and we set $r = 2s$. We shall assume that N is even and define m by $N = 2m$. If $r < N$, $g^N = 0$. The functions g and g^N are said to be *extreme* if λ_σ are all equal. The above definitions may be easily generalized to apply to ensembles.

II. THE AGP 2-MATRIX

It was proved by Blatt[9] that the BCS ansatz is equivalent to taking a wavefunction of the form

$$\Psi = B^m\Psi_0, \tag{2.1}$$

where $m = \frac{1}{2}N$, Ψ_0 is the vacuum and

$$B = \sum \xi_i a_{2i-1}a_{2i} \tag{2.2}$$

creates an arbitrary geminal (i.e., a two-particle function). In configuration space, Ψ is simply (7.1) of SFDM-I:

$$\begin{aligned}\Psi &= g^N(12\cdots N)\\ &= c_NA_N[g(12)g(34)\cdots g(N-1,N)],\end{aligned} \tag{2.3}$$

where g is an arbitrary normalized geminal, A_N is the antisymmetrizer, and c_N is a normalization constant.

In the theory of exterior forms or Grassmann algebras such an expression is simply a power under exterior multiplication, whence our name *antisymmetrized geminal power* (AGP). If α_i are the orthonormal orbitals of g with eigenvalues λ_i, then, for $g(12) = -g(21)$ the eigenvalues are evenly degenerate, so that

$$g(12) = \sum \xi_i[\alpha_{2i-1}(1)\alpha_{2i}(2) - \alpha_{2i-1}(2)\alpha_{2i}(1)], \tag{2.4}$$

where $|\xi_i|^2 = \lambda_{2i-1} = \lambda_{2i}$.

To simplify our notation, we allow σ or τ to represent *pairs* of natural numbers consisting of an odd number and its successor. For $\sigma = \{2i-1, 2i\}$, let

$$[\sigma] = (2)^{-\frac{1}{2}}[\alpha_{2i-1}(1)\alpha_{2i}(2) - \alpha_{2i-1}(2)\alpha_{2i}(1)] \tag{2.5}$$

[7] P. Nozières, *The Theory of Interacting Fermion Systems* (W. A. Benjamin, Inc., New York, 1964).

[8] A. J. Coleman, Can. J. Phys. **42**, 226 (1964).

[9] J. M. Blatt, Progr. Theoret. Phys. (Kyoto) **23**, 447 (1960). Also see K. Nakamura, *ibid.* **21**, 273 (1959).

and

$$\xi_\sigma = \xi_i; \qquad |\xi_\sigma|^2 = \lambda_\sigma = \lambda_{2i-1} = \lambda_{2i}. \tag{2.6}$$

Since g is normalized,

$$\sum \lambda_i = 2 \sum \lambda_\sigma = 1 \tag{2.7}$$

and Eq. (2.4) may be abbreviated to

$$g = \sqrt{2} \sum \xi_\sigma[\sigma]. \tag{2.8}$$

If the rank of g, $r = 2s$, is finite, σ assumes s distinct values. For N even, $N = 2m$, it is easily seen that

$$g^N = (a_m)^{-\frac{1}{2}} \sum c(\sigma_1\sigma_2 \cdots \sigma_m)[\sigma_1\sigma_2 \cdots \sigma_m], \tag{2.9}$$

where $c(\sigma_1 \cdots \sigma_m) = \prod \xi_\sigma$ for $\sigma \in \{\sigma_1, \sigma_2, \cdots \sigma_m\}$; the summation is on distinct m-tuples of σ which we assume are ordered ($\sigma_1 < \sigma_2$ if $i \in \sigma_1$, $j \in \sigma_2$ with $i < j$); $[\sigma_1\sigma_2 \cdots \sigma_m]$ is a normalized Slater determinant; and a_m is the usual notation for the symmetric function of λ_σ, of weight m, defined by the generating function

$$\prod (1 + \lambda_\sigma t) = \sum a_m t^m. \tag{2.10}$$

For example, $a_1 = \sum \lambda_\sigma$, $a_2 = \sum_{\sigma<\tau}\lambda_\sigma\lambda_\tau$.

By Theorem 3.2 of SFDM-I, the 2-matrix of g^N, $D^2(g^N\,|12; 1'2')$, may be expanded in terms of α_i and therefore in terms of $[kl]$ and $[k'l']^*$, where

$$\sqrt{2}\,[kl] = \alpha_k(1)\alpha_l(2) - \alpha_k(2)\alpha_l(1), \tag{2.11}$$

$$\sqrt{2}\,[k'l'] = \alpha_k(1')\alpha_l(2') - \alpha_k(2')\alpha_l(1'). \tag{2.12}$$

If the expression (2.9) for g^N is substituted in the definition for D^2, by a straightforward calculation which employs the orthonormality of the α_i, we obtain the following result.

Theorem 2.1.

$$D^2(g^N \mid 12; 1'2') = \sum b(ijkl)[ij][k'l']^*, \tag{2.13}$$

where the summation is on $i < j$, and $k < l$ and where $b(ijkl)$ is zero except if

(a) $[ij] = [kl] = [\sigma]$; $b(\sigma\sigma) = 2c\lambda_\sigma a_{m-1}(\hat\sigma)$, (2.14a)

(b) $\sigma \neq \tau : b(\sigma\tau) = 2c\xi_\sigma\xi_\tau^* a_{m-1}(\hat\sigma\hat\tau)$, (2.14b)

(c) $\sigma \neq \tau$, $i \in \sigma$, $j \in \tau : b(ijij)$
$= 2c\lambda_\sigma\lambda_\tau a_{m-2}(\hat\sigma\hat\tau)$, (2.14c)

with $c = [N(N-1)a_m]^{-1}$. The caret on $\hat\tau$ indicates that λ_τ is omitted in the product (2.10) defining a_m, that is

$$a_{m-1}(\hat\sigma) = \partial a_m/\partial\lambda_\sigma \quad \text{and} \quad a_{m-2}(\hat\sigma\hat\tau) = \partial^2 a_m/\partial\lambda_\sigma\,\partial\lambda_\tau.$$

It follows from Theorem 2.1 that for g of finite rank, $r = 2s$, with respect to the basis $[ij]$, D^2 is a partitioned matrix

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

in which D_{11} is the $s \times s$ matrix given by (2.14a) and (2.14b), $D_{12} = D_{21} = 0$, and D_{22} is the $2s(s-1) \times 2s(s-1)$ *diagonal* matrix whose non-zero terms are given by (2.14c). The basis $[ij]$ has $s(2s-1)$ terms.

It is instructive to check the coefficients (2.14a) and (2.14c) by evaluating Tr (D^2). Note that each coefficient (2.14c) occurs *four* times. For example if $\sigma = \{1, 2\}$, $\tau = \{3, 4\}$ then $b(1313) = b(1414) = b(2323) = b(2424)$. Thus Tr (D^2) is $2c$ times

$$\sum \lambda_\sigma a_{m-1}(\hat\sigma) + 4 \sum_{\sigma<\tau} \lambda_\sigma\lambda_\tau a_{m-2}(\hat\sigma\hat\tau). \tag{2.15}$$

But

$$\sum \lambda_\sigma a_{m-1}(\hat\sigma) = m a_m \quad \text{and} \quad 2 \sum_{\sigma<\tau} \lambda_\sigma\lambda_\tau a_{m-2}(\hat\sigma\hat\tau)$$
$$= \sum \lambda_\sigma \sum_{\tau\neq\sigma} \lambda_\tau a_{m-2}(\hat\sigma\hat\tau)$$
$$= (m-1) \sum \lambda_\sigma a_{m-1}(\hat\sigma) = m(m-1)a_m.$$

Thus (2.15) equals $m(2m-1)a_m$ and since $N = 2m$, Tr $D^2(g^N) = c2m(2m-1)a_m = 1$.

The largest eigenvalue of D^2, λ_1^2, is of particular interest. By (7.8) of SFDM-I, we know that for even N greater than 3,

$$(N-1)^{-1}[1 - (N-2)b_N] \le \lambda_1^2 < (N-1)^{-1}, \tag{2.16a}$$

where

$$b_N = \text{Tr}\,[D^1(g)D^1(g^{N-2})]. \tag{2.16b}$$

The first inequality in (2.16a) provides a criterion for the onset of ODLRO to which we shall return in SFDM-III.

We use Theorem 2.1 to obtain a new bound on the eigenvalues of an AGP 2-matrix, which has some consequences to be discussed in Sec. V below and in SFDM-III.

Theorem 2.2. For an AGP system of $N = 2m$ fermions and rank $r = 2s$, suppose that the 2-matrix is partitioned as above then, (i) the average of the eigenvalues of D_{11} is $[s(N-1)]^{-1}$, (ii) $2[N(N-1)]^{-1}$ is an upperbound for the eigenvalues of D_{22} and their average is $(m-1)[s(s-1)(N-1)]^{-1}$, (iii) the *sum* of any m eigenvalues of D_{11} or of D_{22} is bounded by $(N-1)^{-1}$.

Proof. By Theorem 2.1, the sum of the s eigenvalues of D_{11} is $2c \sum \lambda_\sigma a_{m-1}(\hat\sigma) = 2cma_m = (N-1)^{-1}$. Thus, since $m \le s$, (i) and the first half of (iii) follow immediately.

By Theorem 2.1, a typical eigenvalue of D_{22} is $2c\lambda_\sigma\lambda_\tau a_{m-2}(\hat{\sigma}\hat{\tau}) \leq 2ca_m = 2[N(N-1)]^{-1}$. This latter value is attained for Slater-type functions for which $r = N$ and is therefore the best possible bound. Since $N - 2 = 2(m-1)$ and $\mathrm{Tr}\,(D_{22}) = \mathrm{Tr}\,(D^2) - \mathrm{Tr}\,(D_{11}) = 1 - (N-1)^{-1} = (N-2)(N-1)^{-1}$, the average of the $2s(s-1)$ eigenvalues of D_{22} is given by (ii). Using the fact that $m \leq s$, (iii) follows easily from (i) and (ii). Q.E.D.

Theorem 2.2 constitutes, for AGP systems, a considerable sharpening of SFDM-I (6.23) which states that $(N-1)^{-1}$ and N^{-1} are upperbounds for the *individual* eigenvalues of N-representable 2-matrices for N even and odd, respectively. Indeed, suppose one 2-eigenvalue approaches arbitrarily close to $(N-1)^{-1}$, then Theorem 4.2 suggests that s must be large and thus by Theorem 2.2 all other eigenvalues are small. A somewhat similar "coupling" between the values of the pairon occupation numbers was noticed in a different context by Blatt and Matsubara.

III. THE AGP 1-MATRIX

In this section we obtain an explicit expression for the 1-matrix (one-particle reduced density matrix) of an arbitrary AGP system. We adopt the notation of Sec. II.

From Theorem 2.1, setting $2 = 2'$ and integrating, we obtain

$$\begin{aligned}D^1(g^N \mid 1; 1') &= \sum_\sigma c\lambda_\sigma a_{m-1}(\hat{\sigma})P_\sigma \\ &\quad + 2c \sum_\sigma \lambda_\sigma \sum_{\tau\neq\sigma} \lambda_\tau a_{m-2}(\hat{\sigma}\hat{\tau})P_\sigma \\ &= c \sum_\sigma (N-1)\lambda_\sigma a_{m-1}(\hat{\sigma})P_\sigma \\ &= (Na_m)^{-1} \sum \lambda_\sigma a_{m-1}(\hat{\sigma})P_\sigma,\end{aligned}$$

where

$$P_\sigma = \alpha_{2i-1}(1)\alpha^*_{2i-1}(1') + \alpha_{2i}(1)\alpha^*_{2i}(1') \tag{3.1}$$

for $\sigma = \{2i-1, 2i\}$. Since $\mathrm{Tr}\,(P_\sigma) = 2$,

$$\begin{aligned}\mathrm{Tr}\,(D^1) &= (Na_m)^{-1}2 \sum_\sigma \lambda_\sigma a_{m-1}(\hat{\sigma}) \\ &= (Na_m)^{-1}2ma_m = 1.\end{aligned}$$

Theorem 3.1. For the arbitrary geminal g given by (2.4), the 1-matrix, $D^1(g^N \mid 1;1')$, of the AGP function g^N is given by

$$D^1(g^N \mid 1; 1') = \sum_\sigma \lambda^1_\sigma P_\sigma, \tag{3.2}$$

where P_σ is defined by (3.1) and

$$\lambda^1_\sigma = (Na_m)^{-1}\lambda_\sigma a_{m-1}(\hat{\sigma}). \tag{3.3}$$

We recall that a_m and $a_m(\hat{\sigma})$ are defined in Theorem 2.1 and are such that

$$a_{m-1}(\hat{\sigma}) = \partial_\sigma a_m, \tag{3.4}$$

where ∂_σ denotes the partial derivative with respect to λ_σ. Thus

$$\lambda^1_\sigma = N^{-1}\lambda_\sigma\partial_\sigma(\ln a_m). \tag{3.5}$$

The eigenvalues λ^1_σ of $D^1(g^N)$ are related by (3.5) to the corresponding eigenvalues λ_σ of

$$D^1(g) = \sum \lambda_\sigma P_\sigma \tag{3.6}$$

in such a manner that $\lambda_\sigma = \lambda_\tau$ implies $\lambda^1_\sigma = \lambda^1_\tau$. In particular for g^N of *extreme* type all λ_σ, and therefore, all λ^1_σ, are equal, and for the 1-matrices normalized to unity it follows that

$$\lambda_\sigma = \lambda^1_\sigma = r^{-1} \tag{3.7}$$

where r is the rank of g.

Theorem 3.2. Consider (λ^1_σ) as a point in Euclidean space of s dimensions, then as λ_σ varies subject only to $\sum \lambda_\sigma = \frac{1}{2}$, $\lambda_\sigma \geq 0$, the point (λ^1_σ) given by (3.5) describes a hypersurface which we denote by S_N. A necessary and sufficient condition that a 1-matrix be representable by the AGP function g^N is that its eigenvalues be evenly degenerate and, denoting them by λ^1_σ, that the point (λ^1_σ) lie on S_N.

Proof. That the condition is necessary follows from Theorem 3.1. It is also sufficient. For we may denote two orthonormal eigenfunctions of $D^1(\Psi)$ corresponding to λ^1_σ by α_{2i-1} and α_{2i}, then defining g by (2.4) with $\lambda_{2i-1} = \lambda_{2i} = \lambda_\sigma$ it follows that $D^1(g^N) = D^1(\Psi)$. Q.E.D.

For $D^1(\Psi)$ satisfying Theorem 3.2, if λ^1_σ are distinct, then g will be uniquely determined apart from the phases of ξ_i. Thus Theorem 3.2 provides a good illustration of the remark in Sec. 5 of SFDM-I that the N-representability of a 1-matrix can be characterized by conditions on its eigenvalues alone.

IV. EXTREME AGP FUNCTIONS

In SFDM-I, we called an AGP function g^N *extreme* if the eigenvalues of the 1-matrix of g were equal. As noted in Sec. III, the eigenvalues of $D^1(g^N)$ are then also equal, so that in the independent particle picture g^N corresponds to a system of N particles equally distributed over $2s$ orbitals ($2s \geq N$). For $2s = N$ such systems are simply of Slater type, but for $2s > N$ the same 1-matrix arises from many different g and thus, as follows from Theorem 4.2, corresponds to systems which are markedly different from one another even though in the independent particle picture they would be regarded as identical.

Extreme g^N are interesting because it was proved in SFDM-I (i) that $D^2(g^N)$ is an extreme point of the convex set of N-representable 2-matrices if g and therefore g^N are of extreme type,[10] and (ii) following Yang[6] and Sasaki,[11] that the least upper bound on the eigenvalues of N-representable 2-matrices is approached arbitrarily closely only by the largest eigenvalue of extreme AGP functions.

On the other hand, Theorem 4.3 of SFDM-I asserts that $\lambda_1^p \leq \langle \alpha_1^p \alpha_1^q \,|A_N(\alpha_1^p\alpha_1^q)\rangle$, where λ_1^p is the largest eigenvalue of $D^p(\Psi)$, α_1^p and α_1^q are the corresponding eigenfunctions of $D^p(\Psi)$ and $D^q(\Psi)$ and A_N is the antisymmetrizer; *equality* obtains if and only if Ψ is proportional to $A_N(\alpha_1^p\alpha_1^q)$. It is therefore natural to ask for conditions under which an arbitrary geminal g is a natural geminal of g^N. In other words, when is g an eigenfunction of $D^2(g^N)$?

Theorem 4.1. The geminal g is an eigenfunction of $D^2(g^N)$ with nonvanishing eigenvalue if and only if g is of extreme type, that is if the eigenvalues of $D^1(g)$ are all equal.

Proof. Suppose that g is an eigenfunction of $D^2 = D^2(g^N)$ with eigenvalue γ, $\gamma \neq 0$, then

$$D^2 g = \gamma g. \qquad (4.1)$$

With g given by (2.8) we find, using (2.14) that

$$D^2 g = 2\sqrt{2}\, c \sum_\sigma [\lambda_\sigma a_{m-1}(\hat{\sigma}) + \sum_{\tau\neq\sigma} \lambda_\tau a_{m-1}(\hat{\sigma}\hat{\tau})]\xi_\sigma[\sigma]. \qquad (4.2)$$

Substituting in (4.1), equating coefficients of $[\sigma]$ and noting that $\xi_\sigma \neq 0$, we obtain the s equations

$$2c[\lambda_\sigma a_{m-1}(\hat{\sigma}) + \sum_{\tau\neq\sigma} \lambda_\tau a_{m-1}(\hat{\sigma}\hat{\tau})] = \gamma. \qquad (4.3)$$

Arguing as in the lines following Eq. (2.15) and noting that $\lambda_\sigma a_{m+1}(\hat{\sigma}) + a_m(\hat{\sigma}) = a_m$, we see that (4.3) takes the form

$$2c[a_m + (m-1)a_m(\hat{\sigma})] = \gamma. \qquad (4.4)$$

From (4.4) we easily conclude that for $\sigma \neq \tau$,

$$a_m(\hat{\sigma}) = a_m(\hat{\tau}),$$

but this is precisely

$$\lambda_\tau a_{m-1}(\hat{\sigma}\hat{\tau}) + a_m(\hat{\sigma}\hat{\tau}) = \lambda_\sigma a_{m-1}(\hat{\sigma}\hat{\tau}) + a_m(\hat{\sigma}\hat{\tau}),$$

and therefore,

$$\lambda_\tau = \lambda_\sigma.$$

Q.E.D.

When the condition of the theorem holds, using (4.3), and (4.8a, b) below, we obtain a result due originally to Yang.[6]

Corollary 4.1a. When the λ_σ are all equal the eigenvalue in Theorem 4.1 is given by

$$\gamma = (N-1)^{-1}[1 - (N-2)/r]. \qquad (4.5)$$

We may also use the above calculation to obtain a modified form of the important inequality (7.8) of SFDM-I. For g normalized, but otherwise arbitrary, the maximum eigenvalue λ_1^2 of $D^2(g^N)$ is bounded below by $\langle g|D^2 g\rangle$ and thus, by (2.8) and (4.2)

$$\lambda_1^2 \geq 4c \sum_\sigma \lambda_\sigma[\lambda_\sigma a_{m-1}(\hat{\sigma}) + \sum_{\tau\neq\sigma} \lambda_\tau a_{m-1}(\hat{\sigma}\hat{\tau})]$$

or

$$\lambda_1^2 \geq 4c \sum \lambda_\sigma[a_m + (m-1)a_m(\hat{\sigma})].$$

Since $\sum \lambda_\sigma = \frac{1}{2}$ and $a_m(\hat{\sigma}) = a_m - \lambda_\sigma a_{m-1}(\hat{\sigma})$, the last expression equals

$$2mca_m - 4(m-1)c \sum (\lambda_\sigma)^2 a_{m-1}(\hat{\sigma}).$$

According to (3.3),

$$\sum (\lambda_\sigma)^2 a_{m-1}(\hat{\sigma}) = Na_m \sum \lambda_\sigma \lambda_\sigma^1$$

and

$$2 \sum \lambda_\sigma \lambda_\sigma^1 = \mathrm{Tr}\,[D^1(g)D^1(g^N)] = b_{N+2},$$

by (2.16b). Noting that $N(N-1)a_m c = 1$, we finally have

$$(N-1)^{-1}[1 - (N-2)b_{N+2}] \leq \lambda_1^2. \qquad (4.6)$$

For g of extreme type the expression on the left of this inequality attains the *maximum* $(N-1)^{-1}[1 - r^{-1}(N-2)]$ of (4.5). We may thus conclude that, for arbitrary g, $b_{N+2} \geq r^{-1}$, and since the right side of this inequality is independent of N, we also have

$$b_N \geq r^{-1}. \qquad (4.7)$$

The inequality (4.7) may be proved by elementary arguments using only the fact that $2\sum \lambda_\sigma = 1$.

An interesting unsolved question is the extent to which g is determined by g^N or $D^2(g^N)$. If $N = r$, g is almost totally undetermined, for in this case there is only one term in the sum (2.9) and *any* two geminals expressible by means of the same N orbitals lead to the same g^N. On the other hand, if g is extreme and $r > N$, then g is determined to within a phase factor as the eigenfunction of $D^2(g^N)$ corresponding to the largest eigenvalue.

[10] The converse statement, that *all* extreme points of the N-representable 2-matrix space arise from *extreme* g^N, the truth of which was conjectured in SFDM-I, is false as will be proved in SFDM-III.

[11] F. Sasaki, Tech. Note 79, Quantum Chemistry Group, Uppsala, 1962.

The form we gave for $D^2(g^N)$ in Theorem 2.1 was partially diagonalized in the sense that $D_{12} = D_{21} = 0$. However, in the case of extreme g we can go farther and find all the eigenvalues of D^2 explicitly. In this case g is necessarily of *finite* rank $r = 2s$ and $\lambda_\sigma = r^{-1} = \lambda$ say. We find that

$$a_m = \binom{s}{m}\lambda^m, \tag{4.8a}$$

$$a_{m-1}(\hat{\sigma}) = \binom{s-1}{m-1}\lambda^{m-1}, \tag{4.8b}$$

$$a_{m-2}(\hat{\sigma}\hat{\tau}) = \binom{s-2}{m-2}\lambda^{m-2}, \tag{4.8c}$$

so that the *diagonal* terms of D_{11} equal $2m[sN(N-1)]^{-1} = 2[r(N-1)]^{-1}$; the *off-diagonal* terms of D_{11} equal $2m(s-m)[s(s-1)N(N-1)]^{-1} = 2(r-N)[r(r-2)(N-1)]^{-1}$ and the *diagonal* terms of D_{22} equal $2m(m-1)[s(s-1)N(N-1)]^{-1} = 2(N-2)[r(r-2)(N-1)]^{-1}$. The eigenvalues of D^2 consist of the nonzero terms (2.14c) of D_{22} together with the eigenvalues of D_{11}. For $s = 3$, D_{11} has the form

$$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

and characteristic equation

$$0 = \begin{vmatrix} a-x & b & b \\ b & a-x & b \\ b & b & a-x \end{vmatrix} = (a+2b-x)\begin{vmatrix} a-x & b & 1 \\ b & a-x & 1 \\ b & b & 1 \end{vmatrix}$$
$$= (a+2b-x)(a-b-x)^2.$$

The preceding calculation is typical of the general case for which D_{11} has (i) one simple eigenvalue equal to $2[r(N-1)]^{-1} + (s-1)2(r-N)[r(r-2)(N-1)]^{-1} = (N-1)^{-1}[1 - r^{-1}(N-2)]$ and (ii) an eigenvalue of multiplicity $s-1$ equal to $2[r(N-1)]^{-1} - 2(r-N)[r(r-2)(N-1)]^{-1} = 2(N-2)[r(r-2)(N-1)]^{-1}$.

Theorem 4.2. Suppose g^N is of extreme type with $r = 2s$, *then* $D^2(g^N)$ has

(i) 2-rank equal to $\frac{1}{2}r(r-1)$,
(ii) *one* simple largest eigenvalue equal to

$$(N-1)^{-1}[1-(N-2)/r], \tag{4.9a}$$

and (iii) one other eigenvalue, of multiplicity $(2s+1)(s-1)$, equal to

$$[2/r(r-2)](N-2)/(N-1). \tag{4.9b}$$

For $r \gg N$, it follows that the 2-matrix of a wavefunction of extreme type has *one* eigenvalue of order N^{-1}; all others are equal and of order r^{-2}.

It is noteworthy that for g^N of Slater type, for which $r = N$, the eigenvalues of D^2 are all equal to $2[N(N-1)]^{-1}$ which, for fixed N, is the minimum of (4.9a) and the maximum of (4.9b).

For $r > 2N$, the largest eigenvalue (4.9a) is *greater* than $\frac{1}{2}(N-1)^{-1}$ and all others are *less* than $\frac{1}{2}N^{-2}$.

Combining the last two theorems, we easily identify the natural geminals for $D^2(g^N)$ when g is extreme.

Corollary 4.2a. For extreme g, the $\frac{1}{2}r(r-1)$ eigenfunctions of $D^2(g^N)$ span the same linear space as do the $[ij]$ of Theorem 2.1; g is a natural geminal corresponding to the largest eigenvalue λ_1^2 and the other natural geminals, corresponding to the one other eigenvalue, may be taken as any $\frac{1}{2}(r-1)(r-2)$ linearly independent geminals orthogonal to g.

Apart from the trivial case of a single Slater determinant, the Corollary 4.2a provides us with the first completely explicit analysis of a 2-matrix.

It may be easily verified that for extreme g, the following equation is exactly valid,

$$D^2(g^N \mid 12; 1'2') = a[\mu(1;1')\mu(2;2') - \mu(1;2')\mu(2;1')] + bg(12)\bar{g}(1'2'). \tag{4.10}$$

Here, $\mu = D^1(g^N)$, $(r-2)(N-1)a = r(N-2)$, and $(r-2)(N-1)b = r-N$. We have remarked previously[12] that the above equality is equivalent to what Blatt regards as the basic assumption common to the Bardeen–Cooper–Schrieffer, the Bogoliubov, and the Quasi Chemical Equilibrium theories of superconductivity. It seems incredible that the electron system of any real solid is described accurately by so simple a wavefunction as an extreme g^N. However, it *is* for extreme g^N that coherent pairing in one geminal is most intense. Apparently, such pairing is the key to an understanding of superconductivity, so that the success of the above theories suggests that an extreme g^N provides a good model for the crucial properties of the electron system of, at least, some metals near 0^0 K.

V. DISCUSSION

We have dealt only with properties of the reduced density matrices which follow from the AGP form of the wavefunction avoiding any discussion of the energy of the system a topic to which SFDM-III will be largely devoted. Even so, the above theorems have immediate physical consequences and stimulate interesting speculation about condensation phenomena.

Part (iii) of Theorem 2.2 suggests that the antisymmetry of the total wavefunction forces a "cou-

[12] A. J. Coleman, Phys. Rev. Letters **13**, 406 (1964).

pling" of the pairon occupation numbers such that if the occupation of one geminal approaches $(N-1)^{-1}$ all the others approach zero. Blatt and Matsubara[13] found a similar phenomenon in a statistical context in which energy considerations were involved. For over a year, the author conjectured[14] that Theorem 2.2 (iii) was valid for m eigenvalues of *any* N-representable 2-matrix. He has shown that it is true for $N = 4$ and, in general, for λ_i^2 associated with *strongly* orthogonal geminals. However, a counterexample for $N = 6$ has disposed of the conjecture for arbitrary systems. Proofs will be given in SFDM-III. For fixed rank r, it is a well-defined, important but, to the author's knowledge, unsolved problem to evaluate the maximum of $\lambda_1^2 + \lambda_2^2$. Is it less than or equal to $(N-1)^{-1}$ or, as the author expects, is its difference from $(N-1)^{-1}$ of order N^{-2}? In view of Yang's discussion[6] of ODLRO, it would appear that answers to the above questions will aid us greatly in speculations about condensation phenomena as we attempt to go beyond the BCS approximation.

As was briefly noted in Sec. 4, Theorem 4.1 implies that one-particle occupation numbers n_i are quite inadequate to describe the wavefunction of a system of particles in effective interaction. In the *extreme* case, since $D^1(g)$ is completely degenerate, the geminal g of Theorem 4.1 can be chosen in an infinity of ways leading to distinct $D^2(g^N)$ but to identical $D^1(g^N)$. For a realistic Hamiltonian, these different g^N will have different energies. In the usual treatment of BCS theory, the specifications needed, additional to n_i, are contained in the explicit definition of g. Thus in Valatin's discussion[15] of the BCS approach as an extension of Hartree–Fock, a matrix $\mathcal{K}$ is introduced which specifies both the 1-matrix and Valatin's χ which replaces our g. Only in the case of a single Slater determinant for which the n_i are 0 or 1 do they give a complete description of a wavefunction. For any real system n_i are never 1. This had already been noticed in 1955 by P. O. Löwdin who, in private conversation with the author, remarked that he had been much struck by the fact that, in connection with his basic paper[16] on density matrices, when he actually calculated the occupation numbers for the natural spin orbitals of an atomic system none of them were really close to unity. Even though it is known that the ground state of an interacting macroscopic system of fermions has almost zero overlap with a single Slater determinant, much of the discussion of the independent particle model and of quasiparticles is pursued in a context which assumes that the n_i specify the relevant characteristics of the system. This seems highly questionable and suggests that despite their considerable successes these theories have not yet been given a solid logical foundation.

It is interesting to note from Theorem 4.1 that, except for the physically unlikely *extreme* case, the geminal g which generates the AGP function g^N is *not* a natural geminal of the system. Hence, there is a geminal other than g which is occupied more intensely than g. This observation underlies the author's recent criticism[12] of Blatt's interpretation of Schafroth condensation.

The type of mathematics used in this paper and, more generally, appropriate for discussing antisymmetric systems was quite common knowledge among mathematicians in 1900 but subsequently went out of fashion. It was developed in connection with Grassmann algebra, Pfaffian forms, and the Cartan calculus. There is currently a resurgence of interest among mathematicians in the elementary parts of these theories but, to the author's knowledge, there is available no clear exposition of the known material slanted towards the needs of quantum physicists. Possibly the most useful references[17] are to Weber, Slebodziński, Thomas, and Bourbaki.

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[13] J. M. Blatt and T. Matsubara, Progr. Theoret. Phys. (Kyoto) **23**, 451 (1960).
[14] A. J. Coleman, Tech. Note 103, Quantum Chemistry Group, Uppsala, 1963.
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[17] E. V. Weber, *Vorlesungen über das Pfaffsche Problem*, Teubner, Leipzig, 1900; W. Slebodziński, *Formes extérieures, et leurs applications*, Warsaw 1954; J. M. Thomas, *Systems and Roots* (Wm. Byrd Press, Richmond, Virginia, 1962); N. Bourbaki, *Algèbre* (Hermann & Cie, Paris) Chap. III, (since this work is immortal it is neither possible nor necessary to fix its date.).

Reduced Density Matrices of Quantum Gases. III. Hard-Core Potentials

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In two previous papers, the reduced density matrices of quantum gases, in the grand canonical formalism and for suitably restricted interaction potentials, have been shown to be analytic vector-valued functions of the activity in a neighborhood of the origin, to tend in some sense to well-defined limits as the volume of the system becomes infinite, and to satisfy a cluster decomposition property. The same results are extended here by the same methods to a wider class of potentials, including hard-core, and allowing attractive interactions, which were excluded previously.

INTRODUCTION

IN two previous papers[1,2] (hereafter referred to as I and II), we have used a Wiener integral representation to study the reduced density matrices (RDM) of quantum gases, in the grand canonical formalism, both for Maxwell–Boltzmann (MB) and quantum statistics (QS). For particles interacting by a two-body potential satisfying suitable conditions (I, Sec. 1):

(a) The RDM are vector-valued analytic functions of the activity z in a neighborhood of the origin ($z = 0$) (I, Sec. 3 and 6).

(b) The thermodynamical limit exists. More precisely, the RDM for finite volume V tend in some sense to well defined limits as the volume becomes infinite (I, Secs. 4 and 7).

(c) As a consequence of these properties, the virial expansion converges in a neighborhood of the origin.

(d) The RDM satisfy a cluster decomposition property (CP) in the form of an absolute integrability property of the quantum analogues of the Ursell functions (II).

The potential was supposed to be absolutely integrable, which excluded hard cores, and in the case of QS only, purely repulsive.

The present paper extends the above-mentioned results to hard-core potentials. As a consequence, it will be possible to introduce attractive potentials in the QS case. The class of potentials considered is defined more precisely in Sec. 1. We follow I and II closely and treat successively the MB (Secs. 2 and 3) and the QS case (Secs. 4 and 5). In Sec. 2, we show that the Kirkwood–Salzburg (KS) equations can be considered as a linear equation in a Banach space and use this property to prove the analyticity in z near the origin and the existence of the infinite volume limit. In Sec. 3, we obtain bounds on the RDM directly and prove the CP. Sections 4 and 5 deal with the QS case and treat the same points in the same order. The notations (I.$p.q$), (II.$p.q$), and ($p.q$) refer to Eq. q of Sec. p of I, II, and the present paper, respectively.

1. CONDITIONS ON THE POTENTIAL (CF. I, SEC. 1)

We consider a system of identical particles in ν-dimensional Euclidian space R^ν, interacting through a two-body potential ϕ with the following properties.

(a) ϕ is a real function which depends only on the difference $(x - y)$ of the positions x and y of the two interacting particles considered and is a symmetric function of these two variables or an even function of $x - y : \phi(x - y) = \phi(y - x)$.

(b) $\phi(x) = +\infty$ if $|x| \leq a$, i.e., ϕ has a spherically symmetric hard core of radius a. $\phi(x)$ is continuous for $|x| > a$ except possibly on a closed set F of capacity 0 (I, Appendix 1). F is harmless, as in I; it has no physical interest here.

(c) ϕ is absolutely integrable outside the hard core,

$$\int_{|x|\geq a} |\phi(x)|\, dx < +\infty. \tag{1.1}$$

The restrictions on the growth of $|\phi|$ near the core which follow from (c) will be seen later on to be unessential.

(d) In the MB case, there exists a real constant $B \geq 0$ such that, for any system of different points $x_i \in R^\nu$ $(i = 1, \cdots, n)$,

$$\sum_{i<j} \phi(x_i - x_j) \geq -nB. \tag{1.2}$$

This implies in particular that ϕ is bounded from below by $-2B$. One can obtain better bounds on ϕ if ϕ is spherically symmetric.[3] If ϕ never takes

[1] J. Ginibre, J. Math. Phys. 6, 238 (1965).
[2] J. Ginibre, J. Math. Phys. 6, 252 (1965).
[3] D. Ruelle, Lecture notes of the Theoretical Physics Institute, University of Colorado, Summer 1963.

the value $-\infty$, which can be imposed from the beginning, then (1.2) is automatically satisfied if for at least one pair (i, j), $|x_i - x_j| < a$.

(e) In the QS case only (cf. Ref. 4), there exists a real constant $B \geq 0$ such that for any finite sequence $x_i \in R^\nu$, $i = 0, 1, \cdots, n$, such that $|x_i - x_j| \geq a$ for all (i, j)

$$\sum_{i=1}^{n} \phi(x_0 - x_i) \geq -B.$$

A sufficient condition for (e) to hold is that for $|x| \geq a$, $\phi(x)$ be bounded from below by a negative nondecreasing function of $|x|$ which is an integrable function of x. If (c) is satisfied, this happens in particular if there exists $r_0 \geq a$ such that $\phi(x)$ is a monotonous function of $|x|$ for $|x| \geq r_0$ and is bounded from below for $|x| \leq r_0$. Condition (e) implies the following property:

Let X^α $(\alpha = 1, \cdots, m)$ be m families of j_α points. $X^\alpha = (x_1^\alpha, \cdots, x_{j_\alpha}^\alpha)$, such that for all $(\alpha, i) \neq (\beta, j)$, $|x_i^\alpha - x_j^\beta| \geq a$. Then

$$\sum_{\alpha=1}^{m} \sum_{1\leq i<j\leq j_\alpha} \phi(x_i^\alpha - x_j^\alpha) + 2 \sum_{\alpha<\beta} \sum_{i\leq j_\alpha} \sum_{j\leq j_\beta} \phi(x_i^\alpha - x_j^\beta) \geq -2qB, \quad (1.4)$$

where

$$q = \sum_{\alpha=1}^{m} j_\alpha.$$

In particular, if all the j_α are equal to one, (1.4) reduces to (1.2). Therefore, (e) $\Rightarrow$ (d). Condition (1.4) which we need in that form later on, could be imposed independently of (e) with a different constant $B' \leq B$ in the right-hand side. The corresponding changes in what follows would be obvious, and the improvement in the results negligible.

2. LIMIT OF INFINITE VOLUME IN THE MAXWELL-BOLTZMANN CASE

The notations are the same as in I. The Wiener integral representation of $\exp(-\beta H_m)$ and its continuity with respect to its arguments remain true for potentials satisfying (a) (b) (c) (d) (See Appendix 1). The RDM are defined by (I.2.7, 8, 9) and satisfy the KS equations (I.3.6, 7), which after symmetrization can be cast into the form of a linear equation (I.3.11, 12, 13) in the vector space E of sequences of Wiener integrable essentially bounded functionals of m trajectories $(m = 0, 1, \cdots)$. Now however, the operator K is no longer a bounded operator from E_ξ (defined by I.2.10) to itself (see below). We circumvent this difficulty by defining new Banach spaces G_ξ in which K is defined and bounded.

Let G be the complex vector space of sequences of functionals of m trajectories $\varphi = (\varphi(\omega^m), m = 0, 1, \cdots)$ such that for all m, $\varphi(\omega^m)$ is integrable in the sense of $\int P_{x^m,y^m}(d\omega^m)$ [and $\int P_{x^m,x^m}(d\omega^m)$] for almost every (x^m, y^m) (and x^m), and the integral is a Lebesgue-measurable function of (x^m, y^m) (and x^m).

Let $\Delta(\omega)$ be a real positive, translation invariant, Wiener-integrable functional of one trajectory, bounded from below by $\Delta_0 > 0$, and define

$$\Delta(\omega^m) = \prod_{i=1}^{m} \Delta(\omega_i). \quad (2.1)$$

Let $G_\Delta \subset G$ be the subspace of those $\varphi \in G$ for which

$$||\varphi|| = \sup_m \text{ess. sup.} |\varphi(\omega^m)|/\Delta(\omega^m) < +\infty. \quad (2.2)$$

G_Δ is a Banach space with $||\varphi||$ as the norm of φ. G_Δ contains E_ξ for any $\xi \leq \Delta_0$, and the injection $E_\xi \to G_\Delta$ is continuous. Now let $\varphi \in G_\Delta$. From (I.3.13) we get:

$$\begin{aligned}|K\varphi(\omega^m)| &\leq |z|\, e^{2\beta B}\, ||\varphi|| \sum_0^\infty \frac{1}{n!} \\ &\quad \times \int P_{y^n,y^n}(d\bar{\omega}^n)\, dy^n\, |K(\omega, \bar{\omega}^n)|\, \Delta(\omega^{m-1}, \bar{\omega}^n) \\ &\leq |z|\, e^{2\beta B}\, ||\varphi||\, \Delta(\omega^{m-1}) \\ &\quad \times \exp\left\{\int P_{yy}(d\bar{\omega})\, dy\, |K(\omega, \bar{\omega})|\, \Delta(\bar{\omega})\right\}, \end{aligned} \quad (2.3)$$

where ω^{m-1} is obtained from ω^m by removing ω.

Suppose that for some $\xi > 0$

$$\xi \exp\left\{\int P_{yy}(d\bar{\omega})\, dy\, |K(\omega, \bar{\omega})|\, \Delta(\bar{\omega})\right\} \leq \Delta(\omega). \quad (2.4)$$

Then

$$|K\varphi(\omega^m)| \leq (|z|\, e^{2\beta B}/\xi)\, ||\varphi||\, \Delta(\omega^m). \quad (2.5)$$

Therefore, K is a bounded operator from G_Δ to G_Δ (one verifies that K preserves the relevant measurability properties), and its norm satisfies

$$||K|| = k \leq |z|\, e^{2\beta B}/\xi. \quad (2.6)$$

We are thus led to the problem of solving (2.4) for Δ and ξ, with ξ as large as possible. We first majorize the left-hand side of (2.4). Let $\bar{\omega}'$ be the trajectory obtained from $\bar{\omega}$ by the translation $-y$. We note $\bar{\omega} = \bar{\omega}' + y$. For any two trajectories ω_1, ω_2, we define $\omega_1 - \omega_2$ by $(\omega_1 - \omega_2)(t) = \omega_1(t) - \omega_2(t)$.

[4] O. Penrose, J. Math. Phys. 4, 1312 (1963).

For any (continuous)ω, let $V(\omega)$ be the union of the family of spheres of radius a with centers at the points $\omega(t)$ $(0 \le t \le \beta)$ and $v(\omega)$ the volume of $V(\omega)$. From translation invariance of Δ, we get

$$\int dy\, P_{yy}(d\bar\omega)\, |K(\omega, \bar\omega)|\, \Delta(\bar\omega) = \int P_{00}(d\bar\omega')\Delta(\bar\omega') \int dy\, |K(\omega, \bar\omega' + y)|. \quad (2.7)$$

Now

$$\int dy\, |K(\omega, \bar\omega' + y)| = \int dy \left| \exp\left[-\int_0^\beta \phi(\omega(t) - \bar\omega'(t) - y)\, dt \right] - 1 \right|. \quad (2.8)$$

The hard core contributes to this integral if $y \in V(\omega - \bar\omega')$ and its contribution is $v(\omega - \bar\omega')$. One easily obtains (I.A2) a bound for the contribution of the region $y \notin V(\omega - \bar\omega')$. Finally,

$$\int dy\, |K(\omega, \bar\omega' + y)| \le v(\omega - \bar\omega') + A, \quad (2.9)$$

where

$$A = \beta e^{2\beta B} \int_{|x| \ge a} |\phi(x)|\, dx. \quad (2.10)$$

Instead of A one could also use the better bound

$$\beta \int_{|x| \ge a} \phi_+(x)\, dx + \frac{(e^{2\beta B} - 1)}{2B} \int_{|x| \ge a} \phi_-(x)\, dx, \quad (2.11)$$

where

$$\phi_\pm(x) = \max\,[\pm\phi(x), 0]. \quad (2.12)$$

A sufficient condition for (2.4) to hold is, therefore,

$$\xi \exp\left\{ \int P_{00}(d\bar\omega)[v(\omega - \bar\omega) + A]\Delta(\bar\omega) \right\} \le \Delta(\omega). \quad (2.13)$$

Let $\Delta(\omega) = \xi \exp\,[\varphi(\omega)]$. (2.13) becomes

$$\varphi(\omega) \ge \xi \int P_{00}(d\bar\omega)[v(\omega - \bar\omega) + A] \exp\,[\varphi(\bar\omega)]. \quad (2.14)$$

(2.14) implies in particular that $\varphi(\omega) \ge 0$, and therefore, $\Delta_0 \ge \xi$. The right-hand side of (2.14) is not bounded as a function of ω, therefore (2.14) has no solutions of the type $\varphi =$ const. This corresponds to the above-mentioned unboundedness of K operating in E_ξ. We show in Appendix 2 that there exists $\xi_0 > 0$ such that (2.14) has a minimum real positive solution φ_ξ for any ξ such that $0 \le \xi < \xi_0$ and no finite real positive solution for $\xi > \xi_0$. φ_ξ can be obtained by iteration from 0, and is the value for $\xi \in [0, \xi_0[$ of a function of (ξ, ω) analytic in ξ for $|\xi| < \xi_0$. For any $\xi \in [0, \xi_0[$, let G_ξ be the G_Δ corresponding to $\Delta(\omega) = \xi \exp\,[\varphi_\xi(\omega)]$. $\Delta(\omega)$ is an integrable function of ω, and

$$f_\beta(y - x) = \int P_{xy}(d\omega)\Delta(\omega) \quad (2.15)$$

is a (real positive) bounded integrable function of $(y - x)$ (Appendix 2). We write ess. sup $f_\beta(x) = ||f_\beta||_\infty$ and $\int f_\beta(x)\, dx = ||f_\beta||_1$. It then follows from (2.6) and (2.13), that for any z such that

$$|z| < R = \xi \exp\,(-2\beta B), \quad (2.16)$$

Eq. (I.3.11) has a unique solution in G_ξ, which is analytic in z for $|z| < R$. Moreover, from $||\zeta|| \le |z|/\Delta_0 \le |z|/\xi$, we get

$$||\rho_\Lambda|| \ge (1 - k)^{-1}\, |z|/\xi. \quad (2.17)$$

It follows as in I that ρ_Λ as defined by (I.2.9) and (I.3.11) coincide for $|z| < R$. From (2.2), (2.17), and (I.2.8), we get

$$|\rho_\Lambda(x^m, y^m)| \le (1 - k)^{-1} \frac{|z|}{\xi} \int P_{x^m, y^m}(d\omega^m)\Delta(\omega^m). \quad (2.18)$$

Therefore, $\rho_\Lambda(x^m, y^m)$ is a bounded operator in $L^2(\Lambda^m)$, with

$$||\rho_\Lambda(x^m, y^m)||_2 \le (1 - k)^{-1}(|z|/\xi)\, ||f_\beta||_1^m. \quad (2.19)$$

We now consider the limit $V \to \infty$ and show that the results of I, Sec. 4 remain true in the present case. The notations are the same as in I, Sec. 4 except for the replacement of R, δ, δ' by L, l, l', respectively. We consider first

$$||A_L K A_{L+l'} - A_L K A_{L+l}||:$$

Let $\varphi \in G_\xi$. Then

$$|(A_L K A_{L+l'} - A_L K A_{L+l})\varphi(\omega^m)| \le |z|\, e^{2\beta B}\, ||\varphi||\, \Delta(\omega^{m-1})\alpha_L(\omega),$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int dy^n\, P_{y^n, y^n}(d\bar\omega^n)\, |K(\omega, \bar\omega^n)| \quad (2.20)$$

$$\times [\alpha_{L+l'}(\bar\omega^n) - \alpha_{L+l}(\bar\omega^n)]\Delta(\bar\omega^n) \le |z|\, e^{2\beta B}\, ||\varphi||\, \Delta(\omega^m) C_{l,L}(\beta), \quad (2.21)$$

where

$$C_{l,L}(\beta) = \frac{1}{\xi} \sup_\omega \alpha_L(\omega) \times \int dy\, P_{yy}(d\bar\omega)\, |K(\omega, \bar\omega)|\, \Delta(\bar\omega)[1 - \alpha_{L+l}(\bar\omega)]. \quad (2.22)$$

We take $l > 2a$ and split the domain of the $\bar{\omega}$ integration into two parts:

(1) The $\bar{\omega}$ which stay entirely outside $\Lambda_{L+l/2}$. The hard core plays no role and their contribution is bounded by (see I, Appendix 2)

$$\xi^{-1}\beta e^{2\beta B}\int_{|x|>l/2}|\phi|\,dx\int P_{00}(d\bar{\omega})\Delta(\bar{\omega}). \qquad (2.23)$$

(2) The $\bar{\omega}$ which have points inside $\Lambda_{L+l/2}$ belong to $K'(\frac{1}{2}l,\beta)$ (I.A1–A5). The same method that led to (2.8) gives for their contribution the bound

$$\xi^{-1}\sup_{\omega}\alpha_L(\omega)\int_{K'(l/2,\beta)}P_{00}(d\bar{\omega})[v(\omega-\bar{\omega})+A]\Delta(\bar{\omega}). \qquad (2.24)$$

The integration of the term containing $v(\omega - \bar{\omega})$ can even be restricted to $K'(l - a, \beta)$. (2.23) does not depend on L and $\to 0$ as $l \to \infty$. (2.24) depends on L through $\alpha_L(\omega) \times v(\omega - \bar{\omega})$ and is an increasing function of L. We show in Appendix 2 that it $\to 0$ as $l \to \infty$ for $L = L_0 + pl$, where L_0 and p are real positive constants. From this and (2.21) it follows that

$$||A_LKA_{L+l'} - A_LKA_{L+l}|| \le \eta(l, L), \qquad (2.25)$$

where $\eta(l, L)$ is an increasing function of L for fixed l, and such that $\eta(l, L + pl) \to 0$ as $l \to \infty$ for fixed $L > 0$, $p > 0$.

Lemma 1 (I, Sec. 4) and the end of its proof are then valid with the only change that the limit is now in the sense of the G_ξ topology. In contrast to the preceding case (I) and due to the effective L dependence of the bound (2.25), the limit as $l \to \infty$ is no longer uniform in L.

Lemma 2 (I, Sec. 4) holds without change, the proof being modified as follows: The contribution of the trajectories of the classes α and β are bounded respectively by

$$\epsilon(l, L)\left\{\int P_{x^m,y^m}(d\omega^m)\Delta(\omega^m)\right\} \le \epsilon(l, L)\,||f_\beta||_\infty^m \qquad (2.26)$$

and

$$2\,||\rho||\,m\int P_{x^{m-1},y^{m-1}}(d\omega^{m-1})\Delta(\omega^{m-1})$$
$$\times\int_{K'(r,\beta)}P_{xy}(d\omega)\Delta(\omega) \qquad (2.27)$$
$$\le 2\,||\rho||\,m\,||f_\beta||_\infty^{m-1}\int_{K'(r,\beta)}P_{xy}(d\omega)\Delta(\omega). \qquad (2.28)$$

We show in Appendix 2 that the last integral in (2.28) $\to 0$ as $r \to \infty$ uniformly with respect to (x, y). Lemma 2 follows immediately.

Theorem 1 holds with the following change in the proof. The contribution to (I.4.17) of the region $x^m \notin \Lambda_L^m$ is bounded by

$$[2\,||\rho||]V^m\,||\varphi||^2\sup_{y^m,y'^m\in\Lambda^m{}_{L-r}}$$
$$\times\int_{x^m\notin\Lambda^m{}_L}dx^m\,P_{x^m,y^m}(d\omega^m)\Delta(\omega^m)$$
$$\times P_{x^m,y'^m}(d\omega'^m)\Delta(\omega'^m) \qquad (2.29)$$
$$\le [2\,||\rho||\,V^{m/2}\,||\varphi||]^2\,||f_\beta||_\infty^m\,m\,||f_\beta||_1^{m-1}$$
$$\times\sup_{y\in\Lambda_{L-r}}\int_{x\notin\Lambda_L}dx\,P_{xy}(d\omega)\Delta(\omega). \qquad (2.30)$$

Now

$$\sup\cdots\le\int_{|x|\ge r}f_\beta(x)\,dx. \qquad (2.31)$$

f_β is integrable; therefore, the last quantity $\to 0$ as $r \to \infty$. The theorem follows immediately.

Theorem 2 holds with the following change in the proof of the convergence of $(z/V_L)(d/dz)\log_e \equiv \ln Z_L$ to ρ_0: the bound (I.4.23) is replaced by

$$\left|\frac{1}{V_{L+l}}z\frac{d}{dz}\ln Z_{L+l} - \rho_0\right|$$
$$\le \epsilon_1(l, L) + \frac{2}{1-k}\,||f_\beta||_\infty\left[1 - \frac{L^\nu}{(L+l)^\nu}\right]. \qquad (2.32)$$

The theorem and the convergence of the virial expansion follow as in I.

3. BOUNDS ON THE RDM AND CP IN THE MB CASE

We follow II (Sec. 2). The unexplained notations are those of II (Sec. 1). First, for $\xi < \xi_0$:

$$\int|\tilde{\varphi}_X(\omega^n)|\,P_{y^n,y^n}(d\omega^n)\,dy^n \le R_{mn}\Delta(X), \qquad (3.1)$$

where

$$\begin{cases} R_{mn} = n!\xi^{-1}R^{-(m+n-1)}, & \text{for} \quad m > 1. \\ R_{1n} = n!\xi^{-1}e^{-2\beta B}R^{-n}. \end{cases} \qquad (3.2)$$

The proof is the same as in II and in Ref. 5.

From (3.1), (3.2), (II.2.7), (II.2.9), we obtain the bound

$$|\rho(X)| \le (|z|/\xi)(|z|/R)^{m-1}(1 - |z|/R)^{-1}\Delta(X), \qquad (3.3)$$

which is better than (2.17) by a factor $(|z|/R)^{m-1}$. If the potential satisfies the stronger condition (e)

[5] D. Ruelle, Rev. Mod. Phys. 36, 580 (1964).

instead of (d), we can obtain different bounds on ρ from the Mayer–Montroll equations (II.1.30). From

$$\int P_{yy}(d\omega)\, dy\, |K(X, \omega)|\, \Delta(\omega) \leq \int P_{00}(d\omega)\Delta(\omega)\left[\sum_{i=1}^{m} v(\omega - \omega_i) + mA\right], \quad (3.4)$$

we get as in II

$$\int |\tilde{\varphi}_X(\omega^n)|\, P_{y^n, y^n}(d\omega^n)\, dy^n \leq R_{mn} \exp\left[-U_m(X)\right]\Delta(X) \quad (3.5)$$

and

$$|\rho(X)| \leq (|z|/\xi)(|z|/R)^{m-1}(1 - |z|/R)^{-1} \times \exp\left[-U_m(X)\right]\Delta(X), \quad (3.6)$$

where R_{mn} is defined by (3.2). The inequalities (3.4) and therefore (3.5) and (3.6) remain true if A is replaced by the better bound (2.11) [cf. Ref. 4, especially Eq. (7.11)]. For physical, i.e., real positive z and for potentials satisfying (e), we can obtain better bounds directly Ref. 6. From

$$\exp\left[-U_{m+n}(\omega^{m+n})\right] \leq \exp\left[-U_m(\omega^m)\right] \times \exp\left[m\beta B\right] \exp\left[-U_n(\omega^n)\right] \quad (3.7)$$

and from the definitions, we get

$$\rho(\omega^m) \leq \left[z \exp(\beta B)\right]^m \exp\left[-U_m(\omega^m)\right]. \quad (3.8)$$

We come back to potentials satisfying only (a) (b) (c) (d). The ρ defined by (II.2.7,9) and the infinite-volume limit of ρ_Λ (Sec. 2) are identical; the proof is the same as in II.

We now prove the CP. Let

$$h(x) = \sup_{\omega}\left\{\left[\int P_{0x}(d\bar{\omega})[v(\omega - \bar{\omega}) + A]\Delta(\bar{\omega})\right] \times \left[\int P_{00}(d\bar{\omega})[v(\omega - \bar{\omega}) + A]\Delta(\bar{\omega})\right]^{-1}\right\}. \quad (3.9)$$

Obviously $h(0) = 1$. We prove in Appendix 2 that h is a bounded integrable function of x and that

$$h(x) \geq f_\beta(x)/f_\beta(0). \quad (3.10)$$

Then

$$\int |\tilde{\varphi}_X(\omega^n)|\, P_{y^n, y^n + a^n}(d\omega^n)\, dy^n \leq R_{mn}\Delta(X)h(a^n). \quad (3.11)$$

The proof is the same as that of (3.1). From (3.11) and (II.2.22, 27) we get

$$\int |\varphi(\omega^m)|\, P_{x^{m-1}, (x+a)^{m-1}}(d\omega^{m-1})\, dx^{m-1} \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} R_{1, m+n-1}\Delta(\omega)h(a^{m-1}) \quad (3.12)$$

$$\leq (m-1)!\xi^{-1}e^{-2\beta B}R^{-(m-1)} \times (1 - |z|/R)^{-m}h(a^{m-1})\Delta(\omega). \quad (3.13)$$

We integrate over the last trajectory and use (3.10) to obtain a slightly weaker but more symmetrical result. Finally,

$$\int |\bar{\chi}(x^m, x^m + a^m)|\, dx^{m-1} \leq (m-1)!e^{-4\beta B}f_\beta(0)[|z|/(R - |z|)]^m h(a^m). \quad (3.14)$$

As in II, $\bar{\chi}$ is an integrable function of the distances between the pairs of points (x_i, y_i). The decrease as a function of the distances $|a_i| = |y_i - x_i|$ is slower and less explicitly known than in the previous case. The completely integrated form of the CP still holds,

$$\int |\bar{\chi}(x^m, y^m)|\, dx^m\, dy^{m-1} \leq (m-1)!e^{-4\beta B}f_\beta(0)[|z|/(R - |z|)]^m\, ||h||_1^m. \quad (3.15)$$

4. LIMIT OF INFINITE VOLUME IN THE QUANTUM STATISTICS CASE

We follow I, Secs. 5, 6. The potential now satisfies conditions (a), (b), (c), (e). The notations are those of II, Sec. 3. The RDM are defined by

$$\rho_\Lambda(x^m, y^m) = \left(\sum_{\pi} \epsilon^\pi\right)_m \sum_{j^m} \epsilon^{q+m} \int P^{j^m}_{x^m, \pi(y^m)}(d\omega^m)\rho_\Lambda(\omega^\gamma), \quad (4.1)$$

$$\rho_\Lambda(\omega^\gamma) = \frac{1}{Z_\epsilon}\sum_{r=0}^{\infty} z^{q+r} \sum_{\delta(r)} \int d_\epsilon\omega^\delta\, \alpha_\Lambda(\omega^{\gamma+\delta}) \times \exp\left[-U(\omega^{\gamma+\delta})\right], \quad (4.2)$$

$$Z_\epsilon = \sum_{r=0}^{\infty} z^r \sum_{\delta(r)} \int d_\epsilon\omega^\delta\, \alpha_\Lambda(\omega^\delta) \exp\left[-U(\omega^\delta)\right]. \quad (4.3)$$

The bounds given in I, Sec. 5 hold without change. The Kirkwood–Salzburg equations are now (I.6.8), (II.3.38 et seq.):

$$\rho_\Lambda(\omega^\gamma) = \alpha_\Lambda(\omega^\gamma)z^{j_1} \exp\left[-F_1(\omega^\gamma)\right] \times \left\{\sum_{r=0}^{\infty}\sum_{\delta(r)} \int d_\epsilon\omega^\delta\, K(\omega_1, j_1; \omega^\delta)\rho_\Lambda(\omega^{\gamma'+\delta})\right\} \quad \text{for } m > 1, \quad (4.4)$$

[6] J. Groeneveld, Phys. Letters 3, 50 (1962).

$$\rho_\Lambda(\omega_1, j_1) = \alpha_\Lambda(\omega_1)z^{j_1} \exp[-F_1(\omega_1, j_1)] \times \left\{1 + \sum_{r=1}^{\infty} \sum_{\delta(r)} \int d_\epsilon\omega^\delta K(\omega_1, j_1, \omega^\delta)\rho_\Lambda(\omega^\delta)\right\} \quad \text{for} \quad m = 1. \quad (4.5)$$

We now symmetrize these equations as was done previously in the MB case (I, Sec. 3). This was unnecessary for purely repulsive potentials. (1.4) implies

$$\sum_{i=1}^{m} F_i(\omega^\gamma) \geq -2q\beta B, \quad (4.6)$$

where F_i is defined in analogy with F_1 (II.3.39). Let W_i^γ be the set of the (continuous) ω^γ such that $F_i(\omega^\gamma) \geq -2j_i\beta B$, η_i the characteristic function of W_i^γ and $\theta_i = \eta_i/\sum_{i=1}^m \eta_i$. It follows from (4.6) that $\sum_{i=1}^m \theta_i = 1$. We define Π_k and Π in analogy with (I.3.10). Applying Π to both sides of (4.4) gives (I.6.10) with A_Λ and ζ defined by (I.6.11, 12) and where K is now defined by:

$$K\varphi(\omega^\gamma) = \sum_{i=1}^{m} z^{j_i}\Pi_i\left\{\theta_1(\omega^\gamma) \exp[-F_1(\omega^\gamma)] \times \left[\sum_{r=0}^{\infty} \sum_{\delta(r)} \int d_\epsilon\omega^\delta K(\omega_1, j_1; \omega^\delta)\varphi(\omega^{\gamma'+\delta})\right]\right\} \quad \text{for} \quad m > 1. \quad (4.7)$$

$$K\varphi(\omega_1, j_1) = z^{j_1} \exp[-F_1(\omega_1, j_1)] \times \sum_{r=1}^{\infty} \sum_{\delta(r)} \int d_\epsilon\omega^\delta K(\omega_1, j_1; \omega^\delta)\varphi(\omega^\delta). \quad (4.8)$$

K is not a bounded operator from E_ξ to E_ξ (I.5.13). Let G be the complex vector space of sequences of functionals of m trajectories ω^m of respective lengths $j^m\beta : \varphi = (\varphi(\omega^m, j^m), m = 0, 1, \cdots; j_i = 1, \cdots)$ such that for all (m, j^m): $\varphi(\omega^m, j^m)$ is integrable in the sense of $\int P^{j^m}_{x^m,y^m}(d\omega^m)$ [and $\int P^{j^m}_{x^m,x^m}(d\omega^m)$] for almost every (x^m, y^m) (and x^m), and the integral is a Lebesgue measurable function of (x^m, y^m) (and x^m). Let $\Delta(\omega, j)$ be a sequence of real strictly positive translation invariant functionals of one trajectory of length $j\beta$, $j = 1, \cdots$, integrable in the same sense, and let

$$\Delta(\omega^m, j^m) = \prod_{i=1}^{m} \Delta(\omega_i, j_i). \quad (4.9)$$

Let $G_\Delta \subset G$ be the subspace of those $\varphi \in G$ for which

$$||\varphi|| = \sup_{j^m} \text{ess. sup.} \frac{|\varphi(\omega^m, j^m)|}{\Delta(\omega^m, j^m)} < +\infty. \quad (4.10)$$

G_Δ is a Banach space with $||\varphi||$ as the norm of φ. Let $\varphi \in G_\Delta$. Then

$$|K\varphi(\omega^\gamma)| \leq \sup_{j_i} [|z| e^{2\beta B}]^{j_i} ||\varphi|| \Delta(\omega_i^{\gamma'}) \times \exp\left\{\sum_{1}^{\infty} \frac{1}{j} \int P^j_{vv}(d\omega) dy |K(\omega_i, j_i; \omega, j)| \Delta(\omega, j)\right\}, \quad (4.11)$$

where $\omega_i^{\gamma'}$ is obtained from ω^γ by removing (ω_i, j_i) and provided the series in the last factor converges. If this is the case and if, furthermore, for some $\xi > 0$

$$\xi^{j_0} \exp\left\{\sum_{1}^{\infty} \frac{1}{j} \int P^j_{vv}(d\omega) dy |K(\omega_0, j_0; \omega, j)| \Delta(\omega, j)\right\} \leq \Delta(\omega_0, j_0) \quad (4.12)$$

for all (ω_0, j_0), then

$$|K\varphi(\omega^\gamma)| \leq \sup_{j_i} (|z| e^{2\beta B}/\xi)^{j_i} ||\varphi|| \Delta(\omega^\gamma). \quad (4.13)$$

Therefore, K is a bounded operator from G_Δ to G_Δ provided $|z| e^{2\beta B}/\xi \leq 1$ and its norm satisfies

$$||K|| = k \leq |z| e^{2\beta B}/\xi. \quad (4.14)$$

We come back to (4.12). The method of Sec. 2 gives the sufficient condition:

$$\xi^{j_0} \exp\left\{\sum_{1}^{\infty} \int P^j_{00}(d\omega)\Delta(\omega, j) \times \left[\frac{1}{j} v(\omega, \omega_0) + j_0 e^{j\beta B} A\right]\right\} \leq \Delta(\omega_0, j_0), \quad (4.15)$$

where now

(1) $v(\omega, \omega_0)$ is the volume of the union $V(\omega, \omega_0)$ of the family of spheres of radius a with centers at the points $\omega(t + l\beta) - \omega_0(t + l_0\beta)$, with t real, $0 \leq t \leq \beta$, l and l_0 integers, $0 \leq l \leq j - 1$, $0 \leq l_0 \leq j_0 - 1$.

(2) $A = \beta \int |\phi| dx$.

(3) The factor $\exp(j\beta B)$ comes from the use of condition (e), which is essential here.

Let

$$\Delta(\omega, j) = \xi^j \exp[\varphi(\omega, j)], \quad (4.16)$$

(4.15) becomes

$$\varphi(\omega_0, j_0) \geq \sum_{1}^{\infty} \xi^j \int P^j_{00}(d\omega) \exp[\varphi(\omega, j)] \times \left[\frac{1}{j} v(\omega, \omega_0) + j_0 e^{j\beta B} A\right], \quad (4.17)$$

which implies in particular $\varphi(\omega, j) \geq 0$, and therefore, $\Delta(\omega, j) \geq \xi^j$. We show in Appendix 3 that (4.17) has all the properties stated for (2.14) (Sec. 2). For any $\xi \in]0, \xi_0[$, let $\varphi_\xi(\omega, j)$ be the smallest positive solution, and G_ξ the corresponding G_Δ defined by

(4.10) and (4.16) with $\varphi = \varphi_\xi$. It then follows from (4.14) that for any z such that

$$|z| < R = \xi \exp(-2\beta B), \tag{4.18}$$

Eq. (I.6.10) has a unique solution in G_ξ which is analytic in z for $|z| < R$. Moreover, from $||\zeta|| \le |z|/\xi$, we obtain

$$||\rho_\Lambda|| \le (1-k)^{-1}\,|z|/\xi. \tag{4.19}$$

ρ_Λ defined by (I.6.10) and by (4.2) coincide for $|z| < R$ as previously. From (4.1), (4.10), and (4.19), we obtain

$$|\rho_\Lambda(x^m, y^m)| \le m!(1-k)^{-1}(|z|/\xi)\,||g_\xi||_\infty^m, \tag{4.20}$$

where

$$g_\xi(x) = \sum_1^\infty \xi^j \int P_{0x}^j(d\omega)\exp[\varphi_\xi(\omega, j)]. \tag{4.21}$$

g_ξ is shown in Appendix 3 to be a bounded integrable function of x for $\xi < \xi_1$, where ξ_1 is some real number in $]0, \xi_0]$. There is strong evidence, but we have been unable to prove, that one can take $\xi_1 = \xi_0$. From now on, we take $\xi < \xi_1$. $\rho_\Lambda(x^m, y^m)$ is then a bounded operator in $L^2(R^{\nu m})$, with norm:

$$||\rho_\Lambda(x^m, y^m)||_2 \le m!(1-k)^{-1}(|z|/\xi)\,||g_\xi||_1^m. \tag{4.22}$$

We now consider the limit $V \to \infty$ and show that the results of I, Sec. 7 remain true in the present case. We consider first $||A_LKA_{L+l'} - A_LKA_{L+l}||$, where $0 < 2a < l < l'$. Let $\varphi \in G_\xi$. Then

$$\begin{aligned}
&|(A_LKA_{L+l'} - A_LKA_{L+l})\varphi(\omega^\gamma)| \\
&\le \sup_{ji} (|z|\,e^{2\beta B})^{j_i}\,||\varphi||\,\Delta(\omega_i^{\gamma'})\alpha_L(\omega_i) \\
&\times \exp\left\{\sum_1^\infty \frac{1}{j}\int P_{yy}^j(d\omega)\,dy\,|K(\omega_i, j_i; \omega, j)|\,\Delta(\omega, j)\right\} \\
&\times \left\{\sum_1^\infty \frac{1}{j}\int P_{yy}^j(d\omega)\,dy\,|K(\omega_i, j_i; \omega, j)|\,\Delta(\omega, j)\right. \\
&\times \left.[\alpha_{L+l'}(\omega) - \alpha_{L+l}(\omega)]\right\}
\end{aligned} \tag{4.23}$$

$$\le \sup_{ji} (|z|\,e^{2\beta B}/\xi)^{j_i}\,||\varphi||\,\Delta(\omega^\gamma) \times \{\text{same last factor}\}. \tag{4.24}$$

Therefore,

$$||A_LKA_{L+l'} - A_LKA_{L+l}|| \le \eta(l, L), \tag{4.25}$$

where

$$\begin{aligned}
\eta(l, L) = \sup_{(\omega_0, j_0)} &\left\{\left(\frac{|z|}{R}\right)^{j_0} \sum_1^\infty \frac{1}{j}\int P_{yy}^j(d\omega)\,dy\,\alpha_L(\omega_0)\right. \\
&\times \left.[1 - \alpha_{L+l}(\omega)]\,|K(\omega_0, j_0; \omega, j)|\,\Delta(\omega, j)\right\}.
\end{aligned} \tag{4.26}$$

We split the ω integration as previously:

(1) The contribution of the ω which stay outside $\Lambda_{L+l/2}$ is bounded by

$$[R/(e(R-|z|))]\beta \int_{|x|>l/2} |\phi|\,dx \times \left\{\sum_1^\infty e^{j\beta B}\int P_{00}^j(d\omega)\Delta(\omega, j)\right\}, \tag{4.27}$$

where the first factor comes from the relation $\sup_n nx^n \le [e(1-x)]^{-1}$ for $x < 1$. (4.27) does not depend on L and $\to 0$ as $l \to \infty$.

(2) The contribution of the ω which have points inside $\Lambda_{L+l/2}$ is bounded by

$$\begin{aligned}
&[R/e(R-|z|)]A \sum_{j=1}^\infty e^{j\beta B}\int_{K'(l/2, j\beta)} P_{00}^j(d\omega)\Delta(\omega, j) \\
&+ \sup_{\omega_0}\left\{\sum_{j=1}^\infty \frac{1}{j}\int_{K'(l-a, j\beta)} P_{00}^j(d\omega)\alpha_L(\omega_0)v(\omega_0, \omega)\Delta(\omega, j)\right\}.
\end{aligned} \tag{4.28}$$

The first term in (4.28) depends only on l and $\to 0$ as $l \to \infty$ (Appendix 3). The last term depends on L through $\alpha_L(\omega_0)v(\omega, \omega_0)$ and increases with L. We show in (Appendix 3) that it $\to 0$ as $l \to \infty$ for $L = L_0 + pl$, L_0 and p being positive constants.

Lemma 1 (I, Sec. 7) follows immediately. As in Sec. 2, the limit as $l \to \infty$ is not uniform in L.

Lemma 2 holds without change, the proof being modified as follows: the contributions to (I.7.4) of the trajectories of classes α and β are bounded respectively by

$$m!\epsilon(l, L)\,||g_\xi||_\infty^m \tag{4.29}$$

and

$$m!2\,||\rho||\,m\,||g_\xi||_\infty^{m-1} \times \sup_{x,y\in D}\left\{\sum_{j=1}^\infty \int_{K'(r, j\beta)} P_{xy}^j(d\omega)\Delta(\omega, j)\right\}. \tag{4.30}$$

The last factor $\to 0$ as $r \to \infty$ uniformly with respect to (x, y) (Appendix 3), whence, Lemma 2.

Theorem 1 holds with the following change in the proof. The contributions of the region $x^m \notin \Lambda_L^m$ to $||\Delta\rho\varphi||^2$ is bounded by

$$(2\,||\rho||\,V^{m/2}\,||\varphi||\,m!)^2\,||g_\xi||_\infty^m\,m\,||g_\xi||_1^{m-1} \times \left\{\sum_{j=1}^\infty \int_{|x-y|>r} P_{xy}^j(d\omega)\,dx\,\Delta(\omega, j)\right\}. \tag{4.31}$$

The last factor is $\int_{|x|>r} g_\xi(x)\,dx$, and $\to 0$ as $r \to \infty$, because g_ξ is integrable, whence, the theorem.

Theorem 2 and the convergence of the virial expansion hold with the following change in the proof: the bound (I.7.9) is replaced by

$$\left|\frac{1}{V_{L+l}} z \frac{d}{dz} \ln Z_{L+l} - \rho_0\right| \leq \epsilon_1(l, L) + \frac{2}{1-k} \|g_\xi\|_\infty \left(1 - \frac{L^\nu}{(L+l)^\nu}\right). \quad (4.32)$$

5. BOUNDS ON THE RDM AND CP IN THE QS CASE

We follow II, Sec. 4. The potential satisfies (a), (b), (c), (e). The notations are those of II. We first look for bounds of the type:

$$\int |\tilde{\varphi}_X(\omega^\delta)|\, d_+\omega^\delta \leq P(\gamma, \delta)\Delta(\omega^\delta). \quad (5.1)$$

We proceed by induction on $q + r$. From the KS equations (II.3.38), we get the sufficient condition:

$$P(\gamma, \delta) \geq \sup_{k(\gamma_k \geq 1)} e^{2k\beta B} \sum_{\delta' \leq \delta} P(\gamma' + \delta', \delta - \delta') \times \sup_\omega \frac{1}{\Delta(\omega, k)} \int d_+\omega^{\delta'}\, |K(\omega, k; \omega^{\delta'})|\, \Delta(\omega^{\delta'}), \quad (5.2)$$

where $\omega^{\gamma'}$ is obtained from ω^γ by removing (ω, k). We next look for bounds of the type (II.4.6), for which (5.2) gives the sufficient condition

$$P(q, r) \geq \sup_{1 \leq k \leq q} e^{2k\beta B} \sum_{r'=0}^{r} P(q - k + r', r - r') \times \sup_\omega \frac{1}{\Delta(\omega, k)} \sum_{\delta'(r')} \int d_+\omega^{\delta'}\, |K(\omega, k; \omega^{\delta'})|\, \Delta(\omega^{\delta'}). \quad (5.3)$$

From (4.15, 18) we obtain the solution

$$P(q, r) = \xi^{-1} R^{-(q+r-1)}, \quad (5.4)$$

where we have used the starting point $q + r = 1$, $P(1, 0) \leq [\inf \Delta(\omega, 1)]^{-1} \leq \xi^{-1}$. Therefore,

$$\sum_{\delta(r)} \int |\tilde{\varphi}_X(\omega^\delta)|\, d_+\omega^\delta \leq R^{-(q+r-1)} \Delta(X)/\xi. \quad (5.5)$$

from which we get, as in II,

$$|\rho(X)| \leq (|z|/\xi)(|z|/R)^{q-1}(1 - |z|/R)^{-1}\Delta(X) \quad (5.6)$$

which is better than (4.19) by a factor $(|z|/R)^{q-1}$. We can obtain different bounds for ρ from the Mayer–Montroll equations (II.3.41). From

$$\frac{1}{j} \int P^j_{vv}(d\omega)\, dy\, \Delta(\omega, j)\, |K(X; \omega, j)| \leq \int P^j_{00}(d\omega)\Delta(\omega, j)\left\{\sum_{k=1}^{m} \left[\frac{1}{j} v(\omega_k, \omega) + j_k e^{j\beta B} A\right]\right\}, \quad (5.7)$$

we obtain as previously

$$\sum_{\delta(r)} \int |\tilde{\varphi}_X(\omega^\delta)|\, d_+\omega^\delta \leq R^{-(q+r-1)} \exp[-U(X)]\Delta(X)/\xi. \quad (5.8)$$

and

$$|\rho(X)| \leq (|z|/\xi)(|z|/R)^{q-1}(1 - |z|/R)^{-1} \times \exp[-U(X)]\Delta(X). \quad (5.9)$$

For physical, i.e., real positive z, in the case of Bose–Einstein statistics only, we can obtain better bounds directly.[6] From

$$\exp[-U(\omega^{\gamma+\delta})] \leq \exp[-U(\omega^\gamma)] \exp(q\beta B) \exp[-U(\omega^\delta)] \quad (5.10)$$

and from the definitions, we obtain

$$\rho(X) \leq (z \exp \beta B)^q \exp[-U(X)]. \quad (5.11)$$

We now prove the cluster property, namely, the absolute integrability of $\bar{\chi}(x^m, y^m)$ (II.4.20) as a function of the differences of its arguments, by obtaining a finite upper bound for the integral J_m (II.4.24).

Let $\Delta'(\omega, j)$ be a sequence of real strictly positive translation invariant Wiener integrable (in the same sense as previously) functionals of one trajectory of length $j\beta$, $(j = 1, \cdots)$. Let

$$\Delta'(\omega^m, j^m) = \prod_{i=1}^{m} \Delta'(\omega_i, j_i). \quad (5.12)$$

For any trajectory ω of length $k\beta$, let

$$u_j = u_j(\omega, k) = \int P^j_{00}(d\bar{\omega})\left[\frac{1}{j} v(\omega, \bar{\omega}) + kAe^{j\beta B}\right]\Delta'(\bar{\omega}, j), \quad (5.13)$$

$$v_j = v_j(\omega, k) = \int P^j_{0x}(d\bar{\omega})\, dx\, [v(\omega, \bar{\omega}) + jkAe^{j\beta B}]\Delta'(\bar{\omega}, j). \quad (5.14)$$

Let μ be a real positive constant and consider the equation

$$\xi'^k \exp\left\{\sum_1^\infty (u_j(\omega, k) + \mu v_j(\omega, k))\right\} \leq \Delta'(\omega, k), \quad (5.15)$$

where $0 < \xi' < 1$.

For $\mu = 0$, (5.15) reduces to (4.15) and has the solution (ξ, Δ) used previously, for $0 < \xi < \xi_0$. We show in Appendix 3 that for μ positive and small enough, (5.15) has solutions of the same type as (4.15) for $0 \leq \xi' < \xi_0' \leq \xi_0$. Let (ξ', Δ') be such a solution and let $R' = \xi' \exp(-2\beta B)$. The other notations being the same as in (II. Sec. 4), we now prove the following:

Lemma:

$$\sum_{q+r=p}\sum_{\gamma(m,q)}\sum_{\delta(r)}\int d\omega^{\gamma}\, d_{+}\omega^{\delta}\, |\tilde{\varphi}_T(X,Y)| \leq \mu^{-m}R'^{-(p+s-1)}\xi'^{-1}\Delta'(T). \quad (5.16)$$

Proof: We proceed by induction on $p + s$. We first look for bounds of the type

$$\int d\omega^{\gamma}\, d_{+}\omega^{\delta}\, |\tilde{\varphi}_T(X,Y)| \leq P(\theta,\gamma,\delta)\Delta'(T). \quad (5.17)$$

From (5.17) and (II.4.29) we get the sufficient condition

$$P(\theta,\gamma,\delta) \geq \sup_{k(\theta_k\geq 1)} \exp(2k\beta B) \times \sum_{\gamma'\leq\gamma}\sum_{\delta'\leq\delta} P(\theta'+\gamma'+\delta', \gamma-\gamma', \delta-\delta') \times \sup_{\omega}\left\{\frac{1}{\Delta'(\omega,k)} \times \prod_i \frac{1}{\gamma'_i!}(u_i)^{\gamma'_i}\frac{1}{\delta'_i!}(v_i)^{\delta'_i}\right\}. \quad (5.18)$$

We next look for bounds of the type (II.4.33), for which we obtain the sufficient condition:

$$P(s,q,m,r) \geq \sup_{1\leq k\leq s}\exp(2k\beta B)\sum_{r'=0}^{r}\sum_{m'=0}^{m}\sum_{q'=m'}^{q} P(s-k+q'+r', q-q', m-m', r-r') \times \sup_{\omega}\frac{1}{\Delta'(\omega,k)} \times \text{coef of } z^{r'} \text{ in } \exp\left(\sum_1^{\infty} z^i u_i\right) \times \text{coef of } z^{q'} \text{ in } \left(\sum_1^{\infty} z^i v_i\right)^{m'} \Big/ m'!. \quad (5.19)$$

The last step is to look for bounds of the type

$$\sum_{q\geq m, r\geq 0, q+r=p} P(s,q,m,r) \leq \mu^{-m}P(s,p) \quad (5.20)$$

for which (5.19) gives the sufficient condition

$$P(s,p) \geq \sup_{1\leq k\leq s}\exp(2k\beta B) \times \sum_{p'=0}^{p} P(s-k+p', p-p')\sup_{\omega}\frac{1}{\Delta'(\omega,k)} \times \text{coef of } z^{p'} \text{ in } \exp\left[\sum_1^{\infty} z^i(u_i+\mu v_i)\right]. \quad (5.21)$$

From (5.15) and using the starting point of the induction procedure $s = 1, q = r = 0, P(1, 0) \leq \xi^{-1}$, we obtain the solution

$$P(s,p) = R'^{-(p+s-1)}\xi'^{-1}. \quad (5.22)$$

The lemma follows immediately.

In the preceding proof, for the sake of simplicity, we have overlooked the following fact: the transition from (5.18) to (5.19) and from (5.19) to (5.21) involves an interchange of $\sup_{\omega}$ with a summation. Therefore, conditions (5.19) and (5.21) are not sufficient for (5.18) and (5.19), respectively, to hold and the exact solutions of (5.18) and (5.19) may not satisfy (II.4.33) and (5.20). However, (5.21) is a sufficient condition for the lemma to hold, the purpose of the intermediary bounds being only to reach it in several steps. The same remark applies to the proof of (5.5).

We finally prove the CP. We substitute (5.16) in (II.4.38) and obtain for $|z| < R'$

$$J_m \leq m!\,(m-1)!\,\mu^{-(m-1)} \times \sum_{k=1}^{\infty}\left\{\int P_{0x}^k(d\omega)\, dx\, \Delta'(\omega,k)\right\}\xi'^{-1} \times \sum_{p=m-1}^{\infty} |z|^{p+k}/R'^{p+k-1}, \quad (5.23)$$

$$J_m \leq m!\,(m-1)!\,\mu^{-(m-1)} \times R'(|z|/R')^{m-1}(1-|z|/R')^{-1}\xi'^{-1} \times \sum_{k=1}^{\infty}(|z|/R')^k\int P_{0x}^k(d\omega)\Delta'(\omega,k)\, dx. \quad (5.24)$$

The last series is uniformly convergent for $|z| \leq R'$ and bounded by $||g'_{\xi'}||_1$, where

$$g'_{\xi'}(x) = \sum_{j=1}^{\infty}\int P_{0x}^j(d\omega)\Delta'(\omega,j) \quad (5.25)$$

provided the last series converges in L^1. Now we show in Appendix 3 that for any $\xi < \xi_0$ such that g_ξ be integrable (a sufficient condition for which is $\xi < \xi_1$), for any $\xi' < \xi$, one can take μ sufficiently small for (5.15) to have a solution and for (5.25) to be convergent in L^1. It then follows from (5.24) that the CP holds in the whole region $|z| < R_0 = \xi_0 \exp(-2\beta B)$, $|z| < R_1 = \xi_1 \exp(-2\beta B)$, where we have proved the convergence of the z expansion of the RDM and the convergence in L^1 of the g_ξ series (4.21) for $\xi = |z|\, e^{2\beta B}$. The second condition is very likely to be redundant.

6. CONCLUSIONS

The results of I and II, namely, (1) the existence of the limit $V \to \infty$ in the sense of I, Sec. 4, Lemma

1, 2 and Theorem 1; (2) the analyticity of the RDM in z in a neighborhood of the origin, and therefore, the convergence of the virial expansion in a neighborhood of the origin; (3) the cluster property, remain true for the class of hard-core potentials defined in Sec. 1.

Minor improvements could easily be obtained for the various bounds which appear in the preceding sections; they do not change our main results and are of little practical interest, in view of the fact that most of these bounds contain unknown functions like f, Δ, or g_ξ.

On the other hand, it is clear from the proofs that the same results hold for more general potentials:

In the MB case, condition (e) is never used, except to obtain the bounds (3.6) and (3.8). Keeping (a), (d), and (b) for simplicity, we can replace (c) by the weaker condition:

(c′) ϕ_- is integrable in the whole space, and ϕ_+ is integrable outside a bounded region.

$$\int_{|x|\geq b>a} \phi_+(x)\,dx < +\infty. \tag{6.1}$$

The contribution of $\phi_+(x)$ for $|x| \leq b$ to the various kernels is bounded by that of a hard core of radius b, and all the proofs remain valid with b replacing a. This allows in particular an arbitrary growth of ϕ near the core, as well as potentials without hard core ($a = 0$) and with arbitrary growth at the origin, for instance Lennard–Jones potentials $\phi(x) \simeq |x|^{-12}$.

In the QS case: (c) can also be replaced by (c′), thus allowing an arbitrary growth of ϕ near the core. However, condition (e), and therefore, the existence of a hard core of finite radius is essential to our treatment of non purely repulsive potentials [see Eq. (4.15)] and there is little hope to get rid of it with the present method.

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APPENDIX 1

Integrability of $\exp[-U_m(\omega^m)]$

We first prove that under conditions (a), (b), (d)

$$f(\omega^m) = \exp[-U_m(\omega^m)], \tag{A1.1}$$

where $U_m(\omega^m)$ is defined by (I.2.1), is an integrable function of ω^m for almost every x^m, y^m. The measures referred to in this section are P_{x^m,y^m} in ω space and Lebesgue measure in x space. Let

$$F_m = \{x^m : (x_i - x_j) \in F \text{ for at least one pair } (i,j)\},$$

C_m (resp $\bar{C}_m$) $= \{x^m : |x_i - x_j| < a$ (resp $\leq a$) for at least one pair $(i, j)\}$. Now,

(1) If x^m or $y^m \in C_m$, then $f(\omega^m) = 0$ almost everywhere and

$$\int P_{x^m,y^m}(d\omega^m) f(\omega^m) = 0.$$

(2) The set $\{(x^m, y^m) : x^m \text{ or } y^m \in \bar{C}_m - C_m \text{ or } \in F_m\}$ has measure zero.

(3) Consider now $x^m \notin F_m$, $x^m \notin \bar{C}_m$, $y^m \notin F_m$, $y^m \notin \bar{C}_m$ and define the following sequence of functionals $f_n(\omega^m)$:

(i) If ω^m intersects C_m, $f_n(\omega^m) = f(\omega^m) = 0$.

(ii) If ω^m does not intersect $\bar{C}_m$ and does not intersect F_m,

$$f_n(\omega^m) = \exp\left\{-\frac{\beta}{n}\sum_{p=1}^{n}\sum_{i<j}\phi\left[\omega_i\left(\frac{p\beta}{n}\right) - \omega_j\left(\frac{p\beta}{n}\right)\right]\right\}. \tag{A1.2}$$

The ω^m that intersect F_m form a set of measure 0 (I, Appendix 1). The ω^m that intersect $\bar{C}_m$ but not C_m form also a set of measure 0. We omit the details of the proof. Therefore, $f_n(\omega^m) \to f(\omega^m)$ everywhere and is bounded by $\exp(2m\beta B)$. It then follows from Lebesgue-bounded convergence theorem [Ref. 7] that $f(\omega^m)$ is integrable.

Continuity of $\bar{f}(x^m, y^m) = \int P_{x^m,y^m}(d\omega^m)\alpha_\Lambda(\omega^m) f(\omega^m)$

We prove next that $\bar{f}(x^m, y^m)$ is a continuous function of (x^m, y^m) for $x^m \in \Lambda^m$, $x^m \notin F_m$, $y^m \in \Lambda^m$, $y^m \notin F_m$, provided we define $\bar{f}(x^m, y^m) = 0$ for x^m or $y^m \in \bar{C}_m - C_m$.

The proof is the same as in I, Appendix 1 for x^m and $y^m \in \Lambda^m - \bar{C}_m - F_m$. The only point left to be proved is that for x^m, $y^m \in \Lambda^m - \bar{C}_m - F_m$, $\bar{f}(x^m, y^m) \to 0$ when for at least one pair (i, j), $|x_i - x_j| \to a$. Because $f(\omega^m)$ is bounded [by $\exp(2m\beta B)$], it is sufficient to prove that when $|x - x'| \to a$

$$J_1 = \int_{|\omega-\omega'|\geq a} P^\beta_{xy}(d\omega) P^\beta_{x'y'}(d\omega') \to 0, \tag{A1.3}$$

where the integration is restricted to trajectories ω, ω' such that $|\omega(t) - \omega'(t)| \geq a$ for all $t \in [0, \beta]$.

[7] P. R. Halmos, *Measure Theory* (D. Van Nostrand Company, Inc., New York, 1950).

Let $|x - x'| = a + \epsilon$ and $0 < \gamma < \beta$. We obtain an upper bound for J_1 by replacing the above condition by the weaker one: $|\omega(t) - \omega'(t)| \geq a$ for all $t \in [0, \gamma]$. Therefore,

$$J_1 \leq \int_{|\omega-\omega'|\geq a} P^{\gamma}_{xu}(d\omega)\, du\, P^{\gamma}_{x'u'}(d\omega')\, du' \times \psi_{\beta-\gamma}(y - u)\psi_{\beta-\gamma}(y' - u') \tag{A1.4}$$

$$\leq [\pi(\beta - \gamma)]^{-\nu} J_2, \tag{A1.5}$$

where

$$J_2 = \int_{|\omega-\omega'|\geq a} P^{\gamma}_{xu}(d\omega)\, du\, P^{\gamma}_{x'u'}(d\omega')\, du'. \tag{A1.6}$$

Let

$$\eta = \gamma/n \begin{cases} x_j = \omega(j\gamma/n), & x_0 = x, \quad x_n = u, \\ x_j' = \omega'(j\gamma/n), & x_0' = x', \quad x_n' = u'. \end{cases} \tag{A1.7}$$

Then,

$$J_2 = \lim_{n\to\infty} \int_{|x_j-x_j'|\geq a} \prod_{j=1}^{n} [dx_j\, dx_j'\, \psi_\eta(x_j - x_{j-1})\psi_\eta(x_j' - x_{j-1}')] \tag{A1.8}$$

$$= \lim_{n\to\infty} \int_{|x_j-x_j'|\geq a} \prod_{j=1}^{n} \{d(x_j - x_j')d(x_j + x_j')\psi_{2\eta}[x_j - x_j' - (x_{j-1} - x_{j-1}')]\psi_{2\eta}(x_j + x_j' - x_{j-1} - x_{j-1}')\} \tag{A1.9}$$

$$= \lim_{n\to\infty} \int_{|u_j|\geq a} \prod_{j=1}^{n} [du_j\, \psi_{2\eta}(u_j - u_j')], \tag{A1.10}$$

$$J_2 = \int_{|\omega|\geq a} P^{2\gamma}_{x-x',u}(d\omega)\, du, \tag{A1.11}$$

where ω is now restricted to $|\omega(t)| \geq a$ for all $t \in [0, 2\gamma]$. J_2 is obviously a decreasing function of γ. We now take $x' = (0, \cdots, 0)$, $x = (a + \epsilon, 0, \cdots, 0)$. Let $0 < r < a$ and $0 < \gamma' < 2\gamma$. We obtain an upper bound for J_2 by replacing the above restriction by the weaker one: If the component of $\omega(t)$ perpendicular to $0x$ satisfies $|\omega_\perp(t)| \leq r$ for all $t \in [0, \gamma']$, then the component along $0x$ satisfies $\omega_x(t) > a - \epsilon$ for all $t \in [0, \gamma']$, provided $r^2 \leq \epsilon(2a - \epsilon)$. Therefore,

$$J_2 \leq P_x(K'(r, \gamma')) + \int_{\omega_x(t)\geq a-\epsilon} P^{\gamma'}_{a+\epsilon,u_x}(d\omega_x)\, du_x. \tag{A1.12}$$

Now $P_x(K'(r, \gamma')) \leq 2\sigma(\frac{1}{2}r, \gamma') \to 0$ if $r^2/\gamma' \to \infty$ [Ref. 8, Appendix 1]. The second term is easily seen to be the one dimensional integral $\int_0^\infty du\, [\psi_\gamma(u - 2\epsilon) - \psi_\gamma(u + 2\epsilon)]$ and $\to 0$ if $\epsilon^2/\gamma' \to 0$. We finally choose $r^2 = \epsilon(2a - \epsilon)$ and $\gamma' = \epsilon r$, and see that $J_2 \to 0$ as $\epsilon \to 0$, whence, the result.

Note that the proof rests on the following geometrical property: Any point x_0 of the boundary of the core is the limit as $\epsilon \to 0$ of a point $x(\epsilon)$ which is outside the core, at a distance ϵ from x_0, and such that there exists a disk $D(\epsilon)$ of radius $r(\epsilon)$ with the following properties:

(a) x_0 and $x(\epsilon)$ lie on the axis of $D(\epsilon)$; (b) $D(\epsilon)$ lies entirely inside the core; (c) $d(\epsilon)$ being the distance from $x(\epsilon)$ to $D(\epsilon)$, $d(\epsilon)/r(\epsilon) \to 0$ as $\epsilon \to 0$.

[8] E. Nelson, J. Math. Phys. 5, 332 (1964).

Therefore, if the boundary of Λ is sufficiently smooth, the same method shows that $\bar{f}(x^m, y^m) \to 0$ as one of the x's or one of the y's tends to a point of this boundary. Sufficient conditions are that Λ be convex, or its boundary of class $\mathcal{C}^2$.

Integral Representation of $\exp(-\beta Hm)$

The Wiener integral representation of $\exp(-\beta Hm)$ is obtained as in I. The subdomain of $L^2(\Lambda^m - C_m)$ on which the infinitesimal generator of the semi group T_β is shown to coincide with the usual Hamiltonian is now the class of twice differentiable functions with compact support $\subset \Lambda^m - \bar{C}_m - F_m$.

APPENDIX 2

Solutions of (2.15)

We consider the equations

$$\varphi(\omega) \gtreqless \xi F(\varphi) \equiv \xi \int P_{00}(d\bar{\omega})[v(\omega - \bar{\omega}) + A] \times \exp[\varphi(\bar{\omega})] \tag{A2.1}$$

to which we shall refer as $(\geq)$, $(=)$, $\cdots$ etc. according to the sign standing in the middle. We are interested in solutions of $(\geq)$ with the largest possible $\xi > 0$ and for given ξ, with the smallest possible $\varphi \geq 0$.

We restrict our attention to $\xi \geq 0$, $\varphi \geq 0$. Then,

(1) If $F(\varphi_1)$ exists and $\varphi_1 \geq \varphi_2$, then $F(\varphi_2)$ exists and $F(\varphi_1) \geq F(\varphi_2)$. In particular, $\varphi \geq 0$ implies

$F(\varphi) \geq (v_0 + A)/\lambda^\nu$, where v_0 is the volume of the hard core.

(2) If ($\xi > 0$, $\varphi_0 > 0$) is a solution of ($\geq$), then the sequence φ_0, $\varphi_n = \xi F(\varphi_{n-1})$ is decreasing, bounded from below by $\xi(v_0 + A)/\lambda^\nu$ and, therefore, converges simply to some φ ($\xi(v_0 + A)/\lambda^\nu \leq \varphi(\omega) \leq \varphi_0(\omega)$) which is a solution of (=). $\varphi \leq \varphi_0$ means that φ is better than φ_0 for our purpose.

(3) If ($\xi > 0$, $\varphi > 0$) is a solution of ($\geq$), then (ξ', φ) is also a solution of ($\geq$) for $0 \leq \xi' \leq \xi$. Therefore, (=) has a solution for any $\xi' \leq \xi$.

(4) For ξ sufficiently large, ($\geq$) has no solution. In fact, ($\geq$) implies

$$\xi \leq \frac{\varphi(\omega)}{F(\varphi)} \leq \frac{\lambda^\nu}{v_0 + A} \frac{\operatorname{Inf} \varphi}{\exp(\operatorname{Inf} \varphi)} \leq \frac{\lambda^\nu}{(v_0 + A)e}. \qquad (A2.2)$$

Therefore, there exists $\xi_0 \geq 0$ such that (=) has a solution (ξ, φ) for any $\xi < \xi_0$ and no solution for $\xi > \xi_0$. We see below that $\xi_0 \neq 0$.

(5) If (ξ, φ_0) is a solution of ($\geq$) ($0 \leq \xi < \xi_0$), then the sequence $\varphi_1 = 0$, $\varphi_n = \xi F(\varphi_{n-1})$ is increasing and bounded by φ_0, therefore, converges to some $\varphi_\xi \leq \varphi_0$, which is a solution of (=). φ_ξ is the smallest possible solution for that given ξ. For if φ' is a positive solution of (=), the preceding sequence is bounded term by term by the sequence $\varphi_1' = \varphi'$, $\varphi_n' = F(\varphi_{n-1}') = \varphi'$. The solution is not unique in general [as a trivial example, take $v(\omega - \bar{\omega}) = 0$, no hard core].

(6) In order to show that $\xi_0 \neq 0$, it is sufficient to find an equation

$$\varphi(\omega) = \xi \int P_{00}(d\bar{\omega}) w(\omega, \bar{\omega}) \exp [\varphi(\bar{\omega})] \qquad (A2.3)$$

with

$$w(\omega, \bar{\omega}) \geq v(\omega - \bar{\omega}) + A, \qquad (A2.4)$$

which has a real positive solution ($\xi > 0$, $\varphi > 0$). (ξ, φ) will then be a solution of ($\geq$), and therefore, $\xi_0 \geq \xi > 0$. We need:

Lemma: There exists a constant $\eta > 0$ such that $v(\omega - \bar{\omega}) \leq \eta v(\omega) v(\bar{\omega})/v_0$.

There is strong evidence, but we have been unable to prove, that one can take $\eta = 1$. Elementary geometrical considerations give easily the ridiculous value $\eta = 72^\nu$. The specific value of η is irrelevant in the argument, and we take for simplicity $\eta = 1$ in what follows. We can then take

$$w(\omega, \bar{\omega}) = v(\omega) v(\bar{\omega})/v_0 + A. \qquad (A2.5)$$

The solutions of (A2.3) are of the form $\varphi(\omega) = \tau v(\omega) + \sigma$, where τ and σ are determined by

$$F(\sigma, \tau, \xi) \equiv \tau_0 - \xi e^\sigma \int P_{00}(d\omega) v(\omega) \exp [\tau v(\omega)] = 0,$$

$$G(\sigma, \tau, \xi) \equiv \sigma - A\xi e^\sigma \int P_{00}(d\omega) \exp [\tau v(\omega)] = 0. \qquad (A2.6)$$

Now let

$$f_{\beta\tau}(x) = \int P_{0x}^\beta(d\omega) \exp [\tau v(\omega)]. \qquad (A2.7)$$

We first show that for given complex τ, for β real positive sufficiently small, $f_{\beta\tau}$ exists and is a bounded integrable function of x. Let

$$K_p' = K'(b, \beta/2p) \quad (p \text{ integer}, p \geq 1), \qquad (A2.8)$$

where $K'(r, \gamma)$ is defined by (I.A1.5). If $\omega \notin K_p'$, ω is contained in the union of the p spheres of radius b with centers at the points $\omega((2n - 1)\beta/2p)$, $n = 1, \cdots, p$. Therefore, $v(\omega) \leq pv$, where v is the volume of a sphere of radius $(a + b)$. From this it follows easily that

$$|f_{\beta\tau}(x)| \leq \psi_\beta(x) + (e^{v|\tau|} - 1) \times \left[\psi_\beta(x) + \sum_1^\infty e^{pv|\tau|} P_{0x}(K_p') \right], \qquad (A2.9)$$

and similarly,

$$\int |f_{\beta\tau}(x)| \, dx \leq 1 + (e^{v|\tau|} - 1) \times \left[1 + \sum_{p=1}^\infty e^{pv|\tau|} P_0(K_p') \right], \qquad (A2.10)$$

where $P_0(K_p') = \int_{\omega \in K_p'} P_{0x}^\beta(d\omega) \, dx$. Now from (I.A1.7)

$$P_{0x}(K_p') \leq (c/\lambda^\nu) 2p\sigma(b/4, \beta/2p), \qquad (A2.11)$$

and from (I.A1.4)

$$\sigma(b/4, \beta/2p) \lesssim (\text{polynomial in } pb^2/8\beta) \times \exp(-pb^2/8\beta). \qquad (A2.12)$$

Therefore, for $\beta < b^2/8v|\tau|$, the series in (A2.9) is absolutely convergent, and $f_{\beta\tau}$ is a bounded function of x ($f_{\beta\tau} \in L^\infty$). Analogous bounds on $P_0(K_p')$ [Ref. 8, especially Appendix 1] show that under the same condition on β, the series in (A2.10) converges and $f_{\beta\tau}$ is an integrable function of x ($f_{\beta\tau} \in L^1$).

We next show that for any complex τ and any real positive β, $f_{\beta\tau}$ exists, $f_{\beta\tau} \in L^1$ and $f_{\beta\tau} \in L^\infty$. For any two trajectories (ω_1, ω_2) of respective lengths β_1 and β_2, and such that $\omega_1(\beta_1) = \omega_2(0)$, we define the trajectory $\omega_1 \cup \omega_2$ of length $\beta_1 + \beta_2$ by

$$\begin{aligned}(\omega_1 \cup \omega_2)(t) &= \omega_1(t) & 0 \leq t \leq \beta_1, \\ &= \omega_2(t - \beta_1) & \beta_1 \leq t \leq \beta_1 + \beta_2.\end{aligned} \qquad (A2.13)$$

Then obviously

$$v(\omega_1 \cup \omega_2) \leq v(\omega_1) + v(\omega_2) - v_0 < v(\omega_1) + v(\omega_2). \qquad \text{(A2.14)}$$

Therefore, for any complex τ,

$$|f_{\beta_1+\beta_2,\tau}| \leq f_{\beta_1,|\tau|} * f_{\beta_2,|\tau|}, \qquad \text{(A2.15)}$$

where * means convolution, and in particular,

$$|f_{\beta,\tau}(x)| \leq f^{*n}_{\beta/n,|\tau|}(x), \qquad \text{(A2.16)}$$

whence,

$$||f_{\beta\tau}||_1 \leq ||f_{\beta/n,|\tau|}||_1^n \qquad \text{(A2.17)}$$

and

$$||f_{\beta\tau}||_\infty \leq ||f_{\beta/n,|\tau|}||_1^{n-1}\, ||f_{\beta/n,|\tau|}||_\infty. \qquad \text{(A2.18)}$$

For any complex τ and real positive β. one can choose n sufficiently large for the right-hand side of (A2.16) to exist, whence, the result. Finally, from

$$\left| \frac{f_{\beta,\tau_1}(x) - f_{\beta,\tau_2}(x)}{\tau_1 - \tau_2} - \int P^\beta_{0x}(d\omega) v(\omega) \exp\,[\tau_2 v(\omega)] \right| \leq |\tau_1 - \tau_2|\, f_{\beta,|\tau_2|+|\tau_1-\tau_2|}(x), \qquad \text{(A2.19)}$$

it follows that $f_{\beta\tau}(x)$ is an entire function of τ for fixed β, x. We now come back to the system (A2.6). F and G are entire functions of ξ, σ, τ and $D(F, G)/D(\sigma, \tau)\,|_{\xi=\sigma=\tau=0} = v_0$. Therefore, (A2.6) defines σ and τ as analytic functions of ξ near $\xi = 0$, with $\sigma(0) = \tau(0) = 0$. For ξ real positive, σ and τ are real positive. Therefore, (A2.3) has a solution for some $\xi > 0$; therefore, $\xi_0 \neq 0$.

(7) We only mention that (A2.1) has a complex solution (ξ, φ) such that for fixed ω, φ is an analytic function of ξ for $|\xi| < \xi_0$, and φ coincide with φ_ξ for $\xi \in [0, \xi_0[$. The proof is elementary and makes use of the existence of an analytic solution of (A2.3) near $\xi = 0$ (cf. Ref. 9).

(8) We finally give some properties of $\varphi_\xi(\omega)$ $(0 \leq \xi < \xi_0)$, $\Delta(\omega) = \xi \exp\,[\varphi_\xi(\omega)]$ and $f_\beta(x)$ (2.15). From (=) and (A2.5),

$$\varphi_\xi(\omega) \leq \tau v(\omega) + \sigma, \qquad \text{(A2.20)}$$

where

$$\tau = (\xi/v_0) \int P_{00}(d\omega) v(\omega) \exp\,[\varphi_\xi(\omega)],$$

$$\sigma = A\xi \int P_{00}(d\omega) \exp\,[\varphi_\xi(\omega)] = A f_\beta(0). \qquad \text{(A2.21)}$$

From (A2.20) and the properties of $f_{\beta,\tau}$, it follows that $\Delta(\omega)$ is an integrable function of ω and that $f_\beta(x)$ is a bounded integrable function of x.

[9] R. Bratu, Bull. Soc. Math. France **42**, 113 (1914).

Limit of Infinite Volume

We first consider [see (2.24)]

$$\int_{K'(l/2,\beta)} P_{00}(d\bar\omega)[v(\omega - \bar\omega) + A]\Delta(\bar\omega)\alpha_L(\omega) \leq \int_{K'(l/2,\beta)} P_{00}(d\bar\omega)\left[\frac{v(\omega)v(\bar\omega)}{v_0} + A\right]\Delta(\bar\omega)\alpha_L(\omega), \qquad \text{(A2.22)}$$

and show that this quantity $\to 0$ as $l \to \infty$ for $L = L_0 + pl$ (L_0 and p positive constants). Now $\alpha_L(\omega)v(\omega) \leq \text{const}\cdot L^\nu$. Therefore, it is sufficient to show that

$$\int_{K'(r,\beta)} P_{00}(d\omega)v(\omega)\Delta(\omega) \to 0 \text{ as } r \to \infty$$

faster than any power of r.

For any $\omega \in K'(r, \beta)$, $v(\omega) \geq \text{const} \times r$. Therefore, it is sufficient to show that for any integer n,

$$I_n(r) = \int_{K'(r,\beta)} P_{00}(d\omega)[v(\omega)]^n\Delta(\omega) \to 0 \quad \text{as} \quad r \to \infty. \qquad \text{(A2.23)}$$

From (A2.20) and the properties of $f_{\beta,\tau}$, it follows that $[v(\omega)]^n\Delta(\omega)$ is integrable. This implies[10] that $I_n(r) \to 0$ as $r \to \infty$.

We now show that $\int_{K'(r,\beta)} P_{xy}(d\omega)\ \Delta(\omega) \to 0$ as $r \to \infty$ uniformly with respect to (x, y). From (A2.20) and (A2.15) we obtain

$$\int_{K'(r,\beta)} P_{xy}(d\omega)\Delta(\omega) \leq 2\xi e^\sigma \int_{K'(r/2,\beta/2)} P^{\beta/2}_{xu}(d\omega)\, du \exp\,[\tau v(\omega)] \times \int P^{\beta/2}_{u,y}(d\omega') \exp\,[\tau v(\omega')] \qquad \text{(A2.24)}$$

$$\leq 2\xi e^\sigma\, ||f_{\beta/2,|\tau|}||_\infty \times \int_{K'(r/2,\beta/2)} P^{\beta/2}_{0u}(d\omega)\, du \exp\,[\tau v(\omega)]. \qquad \text{(A2.25)}$$

The last member is independent of (x, y) and $\to 0$ as $r \to \infty$.[10]

Cluster Property—Properties of $h(x)$ (3.9)

From the symmetry of the relation between the trajectories ω, $\bar\omega$, and $(\omega - \bar\omega)$, we get

$$v_0/v(\bar\omega) \leq v(\omega - \bar\omega)/v(\omega) \leq v(\bar\omega)/v_0. \qquad \text{(A2.26)}$$

Therefore,

$$h(x) \leq \left[\int P_{0x}(d\bar\omega)[v(\bar\omega) + A]\Delta(\bar\omega)\right] \times \left[v_0^2 \int P_{00}(d\bar\omega)[v(\bar\omega)]^{-1}\Delta(\bar\omega)\right]^{-1}. \qquad \text{(A2.27)}$$

[10] See Ref. 7, Theorem 23 H, p. 97.

Therefore, h is integrable and bounded. We now show that $h(x) \geq f_\beta(x)/f_\beta(0)$. We write

$$h(x) = \sup_\omega \left\{ \left[\int P_{0x}(d\bar\omega) \frac{v(\omega - \bar\omega) + A}{v(\omega)} \Delta(\bar\omega) \right] \times \left[\int P_{00}(d\bar\omega) \frac{v(\omega - \bar\omega) + A}{v(\omega)} \Delta(\bar\omega) \right]^{-1} \right\}. \quad \text{(A2.28)}$$

Now take $\omega = m\omega_0$, where ω_0 is fixed and $m > 0$ (ω_0 being not reduced to a single point). When $m \to \infty$, $[v(\omega - \bar\omega) + A]/v(\omega)$, considered as a function of $\bar\omega$ tends to one simply and is bounded by $[v(\bar\omega) + A]/v_0$. Therefore, by the Lebesgue bounded convergence theorem,[7] the numerator and denominator in (A2.28) tend, respectively, to $f_\beta(x)$ and $f_\beta(0)$, whence, the result.

APPENDIX 3

Solutions of (4.17)

We consider the system

$$\varphi(\omega, k) \geq F_k(\xi, \varphi) \equiv \sum_{j=1}^{\infty} \xi^j \int P_{00}^j(d\bar\omega) \times \exp[\varphi(\bar\omega, j)] \left\{ \frac{1}{j} v(\omega, \bar\omega) + kAe^{j\beta B} \right\}, \quad \text{(A3.1)}$$

or more simply, $\varphi \geq F(\xi, \varphi)$.

We restrict our attention to $\xi \geq 0$, $\varphi \geq 0$. With the convention that $\varphi_1 \geq \varphi_2$ means $\varphi_1(\omega, j) \geq \varphi_2(\omega, j)$ for all j and all ω, the remarks (1) to (5) of Appendix 2 hold without change. We turn to (6).

(6) As previously $v(\omega, \bar\omega) \leq v(\omega)v(\bar\omega)/v_0$. (A3.2)

We therefore consider the system

$$\varphi(\omega, k) = \sum_{1}^{\infty} \xi^j \int P_{00}^j(d\bar\omega) \exp[\varphi(\bar\omega, j)] \times \{(jv_0)^{-1} v(\omega)v(\bar\omega) + kAe^{j\beta B}\}, \quad \text{(A3.3)}$$

the solutions of which are of the form $\varphi(\omega, k) = \tau v(\omega) + k\sigma$, where τ and σ are determined by

$$F(\sigma, \tau, \xi) \equiv \tau v_0 - \sum_{j=1}^{\infty} \frac{(\xi e^\sigma)^j}{j} \times \int P_{00}^j(d\bar\omega) v(\bar\omega) \exp[\tau v(\bar\omega)] = 0,$$
$$G(\sigma, \tau, \xi) \equiv \sigma - A \sum_{j=1}^{\infty} (\xi e^{\beta B + \sigma})^j \times \int P_{00}^j(d\bar\omega) \exp[\tau v(\bar\omega)] = 0. \quad \text{(A3.4)}$$

Now, from (A2.14) and (A2.18), for any complex τ,

$$\left| \int P_{00}^j(d\omega) \exp[\tau v(\omega)] \right| = |f_{j\beta,\tau}(0)| \leq ||f_{j\beta,\tau}||_\infty \leq ||f_{\beta,|\tau|}||_\infty \times ||f_{\beta,|\tau|}||_1^{j-1}. \quad \text{(A3.5)}$$

Therefore, the series in (A3.4) converges for $\xi e^\sigma ||f_{\beta,\tau}||_1 < 1$ and $\xi e^{\sigma+\beta B} ||f_{\beta,\tau}||_1 < 1$, respectively. From this and the fact that $f_{j\beta,\tau}$ is an entire function of τ, it follows that $F(\sigma, \tau, \xi)$ and $G(\sigma, \tau, \xi)$ are analytic functions of (σ, τ, ξ) near $(0, 0, 0)$. Furthermore $D(F, G)/D(\sigma, \tau)|_{0,0,0} = v_0$. Therefore, (A3.4) determines σ and τ as analytic functions of ξ near $\xi = 0$, with $\sigma(0) = 0$, $\tau(0) = 0$. For ξ real positive, σ and τ are real positive. Therefore, (A3.3) has a solution for some $\xi > 0$; therefore, $\xi_0 \neq 0$ as previously.

(7) The same remark as in Appendix 2 applies here.

(8) We finally give some properties of the solution $\varphi_\xi(\omega, j)(0 \leq \xi < \xi_0)$, $\Delta(\omega, j)$ [Eq. (4.16)] and g_ξ [Eq. (4.21)].

From (A3.1, 2) we obtain

$$\varphi_\xi(\omega, k) \leq \tau v(\omega) + k\sigma, \quad \text{(A3.6)}$$

where

$$\tau = \frac{1}{v_0} \sum_{j=1}^{\infty} \frac{\xi^j}{j} \int P_{00}^j(d\omega) v(\omega) \exp[\varphi_\xi(\omega, j)],$$
$$\sigma = A \sum_{j=1}^{\infty} (\xi e^{\beta B})^j \int P_{00}^j(d\omega) \exp[\varphi_\xi(\omega, j)]. \quad \text{(A3.7)}$$

From (A3.1), we obtain for any pair $(\omega_1, j_1)(\omega_2, j_2)$

$$\varphi(\omega_1, j_1) \leq \varphi(\omega_1 \cup \omega_2, j_1 + j_2) \leq \varphi(\omega_1, j_1) + \varphi(\omega_2, j_2). \quad \text{(A3.8)}$$

From (A3.6) and the properties of $f_{j\beta,\tau}$, it follows that $\Delta(\omega, j)$ is an integrable function of ω. We define now

$$g_{\xi\xi'}(x) = \sum_{j=1}^{\infty} \xi'^j \int P_{0x}^j(d\omega) \exp[\varphi_\xi(\omega, j)] \quad \text{(A3.9)}$$

and

$$g_\xi(x) = g_{\xi\xi}(x) \quad \text{(A3.10)}$$

which is identical with (4.21).

From the convergence of the series in (A3.3) for $\xi < \xi_0$, it follows that $g_{\xi\xi'}(0)$ exists for $\xi < \xi_0$, $\xi' < \xi_0 \exp(\beta B)$. There is strong evidence, but we have been unable to prove, that under the same conditions, the series (A3.9) converges in L^1 [and therefore in L^∞, due to (A3.8)], from which it would follow that $g_\xi(x)$ is a bounded integrable function of x for $\xi < \xi_0$. From (A3.8) however, it follows that

$$\int P^i_{0x}(d\omega) \exp [\varphi_\xi(\omega, j)]\, dx \leq \left\{\int P^1_{0x}(d\omega) \exp [\varphi_\xi(\omega, 1)]\, dx\right\}^i. \quad \text{(A3.11)}$$

Therefore, the series in (A3.9) converges in L^1 for $\xi < \xi_0$,

$$\xi' < \left\{\int P^1_{0x}(d\omega)\, dx \exp [\varphi_\xi(\omega, 1)]\right\}^{-1}. \quad \text{(A3.12)}$$

Therefore, there exists some $\xi_1 \in]0, \xi_0]$ such that $g_\xi(x)$ be in L^1 and L^∞ for $\xi < \xi_1$.

Limit of Infinite Volume

We first show that the expression (4.28) tends to zero as l tends to infinity for $L = L_0 + pl$, L_0 and p positive constants. As in Appendix 2, it is sufficient to show that the two series

$$\sum_{j=1}^{\infty} e^{j\beta B} \int_{K'(l/2,j\beta)} P^j_{00}(d\omega)\Delta(\omega, j) \quad \text{(A3.13)}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{j} \int_{K'(l-a,j\beta)} P^j_{00}(d\omega)[v(\omega)]^n\Delta(\omega, j) \quad \text{(A3.14)}$$

tend to zero as $l \to \infty$. It is easily seen that both are uniformly convergent with respect to l and that each term of both tends to zero as $l \to \infty$, whence, the result. We now show that

$$\sum_{j=1}^{\infty} \int_{K'(r,j\beta)} P^j_{xy}(d\omega)\Delta(\omega, j)$$

tends to zero as r tends to infinity uniformly with respect to (x, y). Now if $(\omega, j) = (\omega_1, j_1) \cup (\omega_2, j_2)$ lies in $K'(r, j\beta)$, then either (ω_1, j_1) lies in $K'(r/2, j_1\beta)$ or (ω_2, j_2) lies in $K'(r/2, j_2\beta)$. From this and (A3.8), we get

$$\begin{aligned}
&\sum_{j=1}^{\infty} \int_{K'(r,j\beta)} P^j_{xy}(d\omega)\Delta(\omega, j) \\
&\leq \int_{K'(r/2,\beta)} P^1_{xu}(d\omega)\, du\, \Delta(\omega, 1) \\
&\times \left[1 + \sum_{j=1}^{\infty} \int P^j_{uy}(d\omega)\Delta(\omega, j)\right] \\
&+ \int P^1_{xu}(d\omega)\Delta(\omega, 1)\, du \sum_{j=1}^{\infty} \int_{K'(r/2,j\beta)} P^j_{uy}(d\omega)\Delta(\omega, j)
\end{aligned}$$

$$\leq \int_{K'(r/2,\beta)} P^1_{0u}(d\omega)\, du\, \Delta(\omega, 1)[1 + ||g_\xi||_\infty] \quad \text{(A3.15)}$$

$$+ \left|\left|\int P^1_{0x}(d\omega)\Delta(\omega, 1)\right|\right|_\infty \times \left\{\sum_1^{\infty} \int_{K'(r/2,j\beta)} P^j_{0u}(d\omega)\Delta(\omega, j)\, du\right\}. \quad \text{(A3.16)}$$

The last member is easily seen to tend to zero as $r \to \infty$, whence the result.

Cluster Property

We consider the system (5.15) which we write

$$\varphi' \geq F(\xi', \varphi') + \mu G(\xi', \varphi') \quad \text{(A3.17)}$$

with

$$\Delta'(\omega, k) = \xi'^k \exp [\varphi'(\omega, k)] \quad \text{(A3.18)}$$

and obvious notations for F and G. For $\mu = 0$, (A3.17) reduces to (A3.1). [For $\mu > 0$, the same arguments as previously show that (A3.17) has positive solutions for $0 \leq \xi' < \xi_0'$. G is positive; therefore, $\xi_0' \leq \xi_0$.]

We now show that if $\xi < \xi_0$ and if $g_\xi \in L^1$, for any $\xi' < \xi$, (A3.17) has a solution ξ', φ' for μ sufficiently small.

Let ξ, φ_ξ be the solution of $\varphi = F(\xi, \varphi)$.

Then

$$\varphi_\xi \geq F(\xi', \varphi_\xi) + (\xi/\xi' - 1)F(\xi', \varphi_\xi). \quad \text{(A3.19)}$$

If $g_\xi \in L^1$, then $G(\xi', \varphi_\xi)$ exists and satisfies

$$[G(\xi', \varphi_\xi)](\omega, k) \leq av(\omega) + bk, \quad \text{(A3.20)}$$

where a and b are positive constants.

On the other hand, $F(\xi', \varphi_\xi)$ is easily seen to be bounded from below by

$$[F(\xi', \varphi_\xi)](\omega, k) \geq a'v(\omega) + b'k, \quad \text{(A3.21)}$$

with $a' > 0$, $b' > 0$.

Therefore,

$$G(\xi', \varphi_\xi) \leq mF(\xi', \varphi_\xi) \quad \text{(A3.22)}$$

with $m > 0$. Therefore, (A3.17) has a solution for $\mu \leq m^{-1}(\xi/\xi' - 1)$. The solution $(\xi', \varphi'_{\xi'})$ corresponding to the sign "=" in (A3.17) satisfies $\varphi'_{\xi'} \leq \varphi_\xi$. Therefore, the convergence of the g_ξ series (4.21) in L^1 implies the convergence of the $g'_{\xi'}$ series (5.25) in L^1.